Optimal control

Several choices available for stabilizing feedback F: for instance, you can choose different α's in Bass algorithm.

Is there an 'optimal' one? One possible way to formalize this:

Linear-quadratic optimal control

Find $u : [0, \infty) \to \mathbb{R}$ (piecewise C^0, let's say) that minimizes

$$V(u) = \int_0^\infty x^* Q x + u^* R u \, \mathrm{d}t$$

$$\text{s.t. } \dot{x} = Ax + Bu, \; x(0) = x_0, \; \lim_{t \to \infty} x(t) = 0.$$

Minimum energy defined by a quadratic form ($R \succeq 0$, $Q \succeq 0$).

We assume $R \succ 0$: control is never free. Trickier problem otherwise.

Linear-quadratic regulator theorem [Datta, Thm 10.5.1]

A solution follows from calculus of variations principles; here is a self-contained version.

Theorem

Let $Q \succeq 0$, $R \succ 0$, $G = BR^{-1}B^T \succeq 0$. Suppose that there exists $X = X^T$ with

- $A^T X + XA + Q - XGX = 0$,
- $A - GX \prec 0$,

Then, the solution of the minimum problem

$$\min \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) \, \mathrm{d}t,$$
$$\text{s.t. } \dot{x}(t) = Ax(t) + Bu(t), \quad \lim_{t \to \infty} x(t) = 0$$

is $x_0^T X x_0$, attained when $u(t) = -R^{-1} B^T X x(t)$ for all t.

Proof

Note that $A - GX \prec 0$ implies $\lim_{t\to\infty} x(t) = 0$, so this u is admissible.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} x^T X x &= \dot{x}^T X x + x^T X \dot{x} \\
&= (Ax + Bu)^T X x + x^T X (Ax + Bu) \\
&= x^T (A^T X + XA) x + u^T B^T X x + x^T X B u \\
&= x^T (X B R^{-1} B^T X - Q) x + u^T B^T X x + x^T X B u \\
&= \underbrace{(u + R^{-1} B^T X x)^T R (u + R^{-1} B^T X x)}_{\geq 0} - x^T Q x - u^T R u.
\end{aligned}$$

Integrating from 0 to ∞,

$$\int_0^\infty x^T Q x + u^T R u \, \mathrm{d}t \geq x_0^T X x_0 - \underbrace{x(\infty)^T X x(\infty)}_{=0},$$

with equality if $u + R^{-1} B^T X x \equiv 0$.

Riccati equation and subspaces

The equation

$$A^T X + XA + Q - XGX = 0, \quad Q \succeq 0,\ G \succeq 0$$

is called algebraic Riccati equation (ARE). It is an invariant subspace problem in disguise, because it can be rewritten as

$$\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (A - GX).$$

The invariant subspace problem

Given $\mathcal{H} = \begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$, find full-rank $U \in \mathbb{R}^{2n\times n}$, $\mathcal{R} \in \mathbb{R}^{n\times n}$ such that $\mathcal{H}U = U\mathcal{R}$. (Then it follows from the first block that $\mathcal{R} = A - GX$).

Solvability conditions

Solutions of (ARE) $\iff$ n-dimensional invariant subspaces of $\mathcal{H}$ with invertible top block.

If $\mathcal{H}$ has distinct eigenvalues, there are at most $\binom{2n}{n}$ solutions (choose n eigenvalues out of the $2n\dots$); otherwise there may even be an infinite number of them.

Solvability conditions

Does the ARE have a stabilizing solution, i.e., one such that $A - GX \prec 0$?

Two things must happen:

- $\mathcal{H}$ has (at least? exactly?) n eigenvalues in the LHP.
- The associated invariant subspace must be of the form $\mathcal{U} = \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}$, with $X = X^T$.

Our next goal: show that these assumptions hold.

Hamiltonian matrices

Matrices of the form

$$\mathcal{H} = \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}, \quad Q = Q^*,\ G = G^*$$

are called Hamiltonian matrix; they satisfy $J\mathcal{H} = -\mathcal{H}^* J$, where $J = \begin{bmatrix} & I \\ -I & \end{bmatrix}$, i.e., they are skew-self-adjoint with respect to the antisymmetric scalar product defined by J.

Spectral symmetry

If $\mathcal{H}v = \lambda v$, then $(v^* J)\mathcal{H} = (-\overline{\lambda})(v^* J)$: $\Lambda(\mathcal{H})$ is symmetric wrt the imaginary axis.

A similar relation can be proved for Jordan chains: λ and $-\overline{\lambda}$ have Jordan chains of the same size.

Thus, it is sufficient to prove that $\mathcal{H}$ has no pure imaginary eigenvalues to conclude that they split $n : n$ between LHP:RHP.

Solvability conditions

Theorem

Assume $Q \succeq 0$, $G = BR^{-1}B^* \succeq 0$, and (A, B) stabilizable. Then, $\mathcal{H}$ has no eigenvalues with $\operatorname{Re}\lambda = 0$.

Proof (sketch)

Suppose instead $\mathcal{H}\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = \imath\omega\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$; from
$0 = -\operatorname{Re}\left[\begin{smallmatrix} z_2^* & z_1^* \end{smallmatrix}\right]\left[\begin{smallmatrix} A & -G \\ -Q & -A^* \end{smallmatrix}\right]\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = z_2^* G z_2 + z_1^* Q z_1$ follows that $Qz_1 = 0$, $z_2^* B = 0$. Hence $-A^* z_2 = -\imath\omega z_2$, but the last two equations then show that (A, B) is not stabilizable (Popov test).

Hence, $\mathcal{H}$ has n eigenvalues in the LHP and n associated ones in the RHP: it has exactly one stabilizing n-dimensional invariant subspace.

Symmetry of the solution

By the Hamiltonian property, if $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ is the span of the eigenvectors in LHP, then $\begin{bmatrix} U_1^* & U_2^* \end{bmatrix} J = \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix}$ is the span of the (left) eigenvectors in RHP.

Left and right invariant subspaces relative to disjoint eigenvectors are orthogonal $\implies$

$$0 = \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = U_2^* U_1 - U_1^* U_2.$$

Form of the invariant subspace

We know now that there exists a (unique) stable invariant subspace

$$\mathcal{U} = \operatorname{Im}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad U_1, U_2 \in \mathbb{R}^{n\times n}.$$

We would like to show that U_1 is invertible. Then,

$$\mathcal{H}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\mathcal{R}$$

can be rewritten with a different basis for the invariant subspace

$$\mathcal{H}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^{-1} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^{-1}(U_1 \mathcal{R} U_1^{-1}), \quad \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} U_1^{-1} =: \begin{bmatrix} I \\ X \end{bmatrix}.$$

In addition,

$$X^* - X = U_1^{-*}U_2^* - U_2 U_1^{-1} = U_1^{-*}(U_2^* U_1 - U_1^* U_2)U_1^{-1} = 0.$$

Nonsingularity of U_1

Suppose (A, B) stabilizable, $Q \succeq 0$, $G \succeq 0$. Then U_1 is invertible.

Proof For any v such that $U_1 v = 0$,

$$-v^* U_2^* G U_2 v = \begin{bmatrix} v^* U_2^* & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} 0 \\ U_2 v \end{bmatrix} = v^* \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \mathcal{R} v = 0.$$

implies $B^* U_2 v = 0$ and $G U_2 v = 0$. The first block row of

$$\begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} v = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \mathcal{R} v$$

gives $U_1 \mathcal{R} v = 0 \implies \ker U_1$ is $\mathcal{R}$-invariant. If $\ker U_1$ is nontrivial, we can find v, $\lambda \in LHP$ such that $U_1 v = 0$, $\mathcal{R} v = \lambda v$. Now the second block row gives $-A^* U_2 v = \lambda U_2 v$. This (together with $B^* U_2 v = 0$ from above) contradicts stabilizability.

Positive definiteness of the solution

Note that

$$ARE \iff (A - GX)^T X + X(A - GX) + Q + XGX = 0.$$

So X solves the Lyapunov equations

$$\hat{A}^T X + X\hat{A} + \hat{Q} = 0, \quad \hat{A} = A - GX,\ \hat{Q} = Q + XGX.$$

And we know that $\Lambda(\hat{A}) \subset LHP, \hat{Q} \succeq 0 \implies X \succeq 0.$

Moreover, we have also shown that under the same assumptions if we also know that (A, B) controllable then $X \succ 0$.

How to solve Riccati equations

- Newton's method (historically the first option).
- Invariant subspace computation: via unstructured methods (QR), 'semi-structured' methods (Laub trick), or fully structured methods (URV).
- Sign iteration (and variants).