

NOTES ON HOMOLOGICAL ALGEBRA

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CONTENTS

1. Background from category theory	1
2. Categories of modules	4
3. Complexes and resolutions	12
4. Derived functors	16
5. Ext and Tor	20
6. Homological dimension	25
7. Applications of Serre's theorem	29
8. The Koszul complex	34
9. Cohomology of groups	42
10. Hochschild homology and cohomology	49
11. The homotopy category and its exact triangles	57
12. The derived category	64
13. Description in the presence of enough projectives	73
14. Total derived functors	79

1. BACKGROUND FROM CATEGORY THEORY

Definition 1.1. A *category* consists of *objects* as well as *morphisms* between pairs of objects; given two objects A, B of a category $\mathcal{C}$, the morphisms from A to B form a set, denoted by $\text{Hom}(A, B)$. (Notice that in contrast to this we do not impose that the objects of the category form a set.) These are subject to the following constraints.

- (1) For each object A the set $\text{Hom}(A, A)$ contains a distinguished element id_A , the identity morphism of A .
- (2) Given two morphisms $\phi \in \text{Hom}(B, C)$ and $\psi \in \text{Hom}(A, B)$, there exists a canonical morphism $\phi \circ \psi \in \text{Hom}(A, C)$, the composition of ϕ and ψ . The composition of morphisms should satisfy two natural axioms:
 - Given $\phi \in \text{Hom}(A, B)$, one has $\phi \circ \text{id}_A = \text{id}_B \circ \phi = \phi$.
 - (Associativity rule) For $\lambda \in \text{Hom}(A, B)$, $\psi \in \text{Hom}(B, C)$, $\phi \in \text{Hom}(C, D)$ one has $(\phi \circ \psi) \circ \lambda = \phi \circ (\psi \circ \lambda)$.

A morphism $\phi \in \text{Hom}(A, B)$ is an *isomorphism* if there exists $\psi \in \text{Hom}(B, A)$ with $\psi \circ \phi = \text{id}_A$, $\phi \circ \psi = \text{id}_B$; we denote the set of isomorphisms between A and B by $\text{Isom}(A, B)$.

Examples 1.2. In these notes, the main examples we'll consider will be algebraic. Thus we shall consider, for example, the category of groups, abelian groups, rings, or modules over a fixed ring R . In all these examples the morphisms are the homomorphisms between appropriate objects.

Remark 1.3. If the objects themselves form a set, we say that the category is *small*. In this case one can associate an oriented graph to the category by taking objects as vertices and defining an oriented edge between two objects corresponding to each morphism.

In the examples above the categories are not small but if we restrict to some set of objects we obtain small subcategories (in the sense to be defined below).

For small categories it is easy to visualize the contents of the following definition.

Definition 1.4. The *opposite category* $\mathcal{C}^{op}$ of a category $\mathcal{C}$ is “the category with the same objects and arrows reversed”; i.e. for each pair of objects (A, B) of $\mathcal{C}$, there is a canonical bijection between the sets $\text{Hom}(A, B)$ of $\mathcal{C}$ and $\text{Hom}(B, A)$ of $\mathcal{C}^{op}$ preserving the identity morphisms and composition.

Next we consider subcategories.

Definition 1.5. A *subcategory* of a category $\mathcal{C}$ is just a category $\mathcal{D}$ consisting of some objects and some morphisms of $\mathcal{C}$; it is a *full subcategory* if given two objects in $\mathcal{D}$, $\text{Hom}_{\mathcal{D}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$, i.e. all $\mathcal{C}$ -morphisms between A and B are morphisms in $\mathcal{D}$.

Examples 1.6. The category of abelian groups is a full subcategory of the category of groups. Given a ring $R \neq \mathbf{Z}$, the category of R -modules is a subcategory of that of abelian groups, but not a full subcategory.

Now comes the second basic definition of category theory.

Definition 1.7. A (*covariant*) *functor* F between two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ consists of a rule $A \mapsto F(A)$ on objects and a map on sets of morphisms $\text{Hom}(A, B) \rightarrow \text{Hom}(F(A), F(B))$ which sends identity morphisms to identity morphisms and preserves composition. A *contravariant functor* from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a functor from $\mathcal{C}_1$ to $\mathcal{C}_2^{op}$.

Examples 1.8. Here are some examples of functors.

- (1) The identity functor is the functor $\text{id}_{\mathcal{C}}$ on any category $\mathcal{C}$ which leaves all objects and morphisms fixed.
- (2) Other basic examples of functors are obtained by fixing an object A of a category $\mathcal{C}$ and considering the covariant functor $\text{Hom}(A, _)$ (resp. the contravariant functor $\text{Hom}(_, A)$) from $\mathcal{C}$ to the category **Sets** which sends an object B the set $\text{Hom}(A, B)$ (resp. $\text{Hom}(B, A)$) and a morphism $\phi : B \rightarrow C$ to the set-theoretic map $\text{Hom}(A, B) \rightarrow \text{Hom}(A, C)$ (resp. $\text{Hom}(C, A) \rightarrow \text{Hom}(B, A)$) induced by composing with ϕ .
- (3) There are *forgetful functors* defined by forgetting structure. For instance, associating to an R -module the underlying abelian group and to an R -module homomorphism the underlying group homomorphism defines the forgetful functor from the category of R -modules to that of abelian groups.
- (4) On the category Mod_R of R -modules important examples of functors are given by tensor product. Fix an R -module B . The rule

$$A \mapsto A \otimes_R B, \quad (\phi : A_1 \rightarrow A_2) \mapsto (\phi \otimes \text{id}_B : A_1 \otimes B \rightarrow A_2 \otimes B)$$

defines a functor $_ \otimes_R B : \text{Mod}_R \rightarrow \text{Mod}_R$. Similarly, tensoring by a module A on the left gives a functor $A \otimes_R _ : \text{Mod}_R \rightarrow \text{Mod}_R$.

Definition 1.9. If F and G are two functors with same domain $\mathcal{C}_1$ and target $\mathcal{C}_2$, a *morphism of functors* Φ between F and G is a collection of morphisms $\Phi_A : F(A) \rightarrow G(A)$ in $\mathcal{C}_2$ for each object $A \in \mathcal{C}_1$ such that for every morphism $\phi : A \rightarrow B$ in $\mathcal{C}_1$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\Phi_A} & G(A) \\ F(\phi) \downarrow & & \downarrow G(\phi) \\ F(B) & \xrightarrow{\Phi_B} & G(B) \end{array}$$

commutes. The morphism Φ is an isomorphism if each Φ_A is an isomorphism; in this case we shall write $F \cong G$.

Remark 1.10. Given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ one can define (modulo some set-theoretic difficulties) a new category called the *functor category* of the pair $(\mathcal{C}_1, \mathcal{C}_2)$ whose objects are functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ and whose morphisms are morphisms of functors. Here the composition rule for some Φ and Ψ is induced by the composition of the morphisms Φ_A and Ψ_A for each object A in $\mathcal{C}_1$.

We now turn to categories with additional properties, abstracting some properties of categories of modules over some ring.

Definition 1.11. A category $\mathcal{A}$ is *additive* if the following hold: