Population models

→ Model the growth of individuals dep. on diff. factors

Option 1: Model N individuals "separtely" : represent events (birth, death, moving away, ...) ⇝ difficult for large populations

Option 2: Just track the number of individuals

$x(t)$ = "# individuals" → simplif. $x(t) \in \mathbb{R}^+$

$$\begin{cases} \dot{x}(t) = \text{"rate of births - deaths"} \cdot x = \underbrace{(b-d)}_{\text{birth rate / death rate}} x \\ x(0) = x_0 \end{cases}$$

$$\dot{x} = (b-d)x \quad \Rightarrow \quad x(t) = e^{(b-d)t} x_0$$

How to solve an ODE ?

Two simple methods: Explicit Euler, Implicit Euler

Runge-Kutta methods

Linear Multistep methods

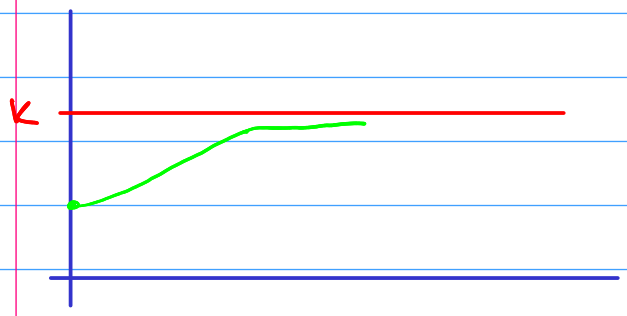
⇝

Idea : Track the availability of natural resources.

$$\dot{x} = a\left(1 - \frac{x}{K}\right) x$$

↳ additional term that track the resources.

Observation If $x_0 = K \Rightarrow \dot{x}(0) = 0 \Rightarrow x(t) \equiv K$ is a solution of the IVP



K = "max number of individuals in the population"

Also in this case the sol. is known: $x(t) = \dfrac{K}{1 + (K/x_0 - 1)e^{-(b-d)t}}$

CHECK $x(0) = x_0$

$\lim_{t \to +\infty} x(t) = \begin{cases} K & b > d \\ 0 & d > b \end{cases}$

Lotka-Volterra: Prey population and Predator population

$x(t)$ = "# prey individuals"

$y(t)$ = "# predators"

$$\begin{cases} \dot{x} = x(a - by) \\ \dot{y} = y(dx - c) \end{cases} \quad a, b, c, d > 0 \text{ parameters}$$

(a: natural growth; by: predators; c: natural death; dx: availability of food)

$$\begin{cases} \dfrac{\dot{x}}{x} = a - by \\ \dfrac{\dot{y}}{y} = dx - c \end{cases}$$

($\dfrac{\dot{x}}{x} = \dot{q}$, $\dfrac{\dot{y}}{y} = \dot{p}$)

Observation: $\dfrac{\dot{x}}{x}$ is the derivative of $\log(x)$

$q = \log x \quad p = \log y$

$$\begin{cases} \dot{q} = a - be^p \overset{?}{=} -\dfrac{\partial H}{\partial p} \\ \dot{p} = de^q - c \overset{?}{=} \dfrac{\partial H}{\partial q} \end{cases}$$ Hamiltonian

$H(q,p) = -ap + be^p - cq + de^q$

EXERCISE: $H(q,p)$ is constant

Proof $\frac{d}{dt} H(q,p) = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}$

$= -\frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0$

Can we find a method that preserves H along the orbits?

YES: Symplectic integrators (Eg. Symplectic Euler)