Population models

Growth w/o constraints

$\dot{x} = ax \rightsquigarrow x(t) = e^{at}$ $\begin{cases} +\infty & a > 0 \\ 0 & a < 0 \end{cases}$ (for $t \to +\infty$)

a ▷ rate of births

Logistic equation

$\dot{x} = a\left(1 - \frac{x}{K}\right)x$

K ▷ limits the pop. to K

$a\left(1 - \frac{x}{K}\right)$ ▷ rate of births

num. analysis: numpy, scipy; plotting: matplotlib; pandas ▷ data analysis

Compartimental models

IDEA Popolation of N individuals

SIR model:

- S: subsceptible
- I: infected
- R: recovered

$\begin{cases} S(t) = \#\text{ subs. individuals} \\ I(t) = \#\text{ infected} \\ R(t) = \#\text{ recovered} \end{cases}$

Every individual transmit the disease at rate $\beta > 0$

Recovery rate $\gamma > 0$

$$\begin{cases} \dot{S} = -\beta SI/N \\ \dot{I} = \beta SI/N - \gamma I \\ \dot{R} = \gamma I \end{cases}$$

rate $\beta \frac{S}{N} I$

Observation: $\dot{S} + \dot{I} + \dot{R} = 0 \Rightarrow S(t) + I(t) + R(t) = N$ is constant.

$\rightsquigarrow$ We expect infections / recovery to happen with $Exp(\lambda)$ distribution

$\sim \frac{1}{\lambda}$ is the average time for an event to happen $\quad \cong$ density $\frac{e^{-\lambda t}}{\lambda}$

To solve it: write it as a single ODE $\begin{cases} \dot{Y} = f(t,Y) \\ Y(t_0) = Y_0 \end{cases}$ $Y \in \mathbb{R}^3$

R_0 parameter = ^{avg}r # individuals infected by a single one

Assume that most individuals are in the "S" class. We have I_0 initial individuals in the I class.

For this population $J(t) \rightsquigarrow \dot{J} = -\gamma J \rightsquigarrow J(t) = e^{-\gamma t} I_0$

The rate of new infections for the $J(t)$ pop. is β $\rightsquigarrow$

$$K = \int_0^{+\infty} J(t)\,\beta\,dt = \int_0^{+\infty} \beta I_0 e^{-\gamma t}\,dt = I_0 \frac{\beta}{\gamma}$$

↳ infections created by the J population

$\rightsquigarrow R_0 = \beta/\gamma$

$R_0 > 1 \Rightarrow$ infection spreads

$R_0 < 1 \Rightarrow$ contraction in # infections

Sometimes we prefer: $i = I/N$, $s = S/N$, $r = R/N$

$s + i + r = 1$