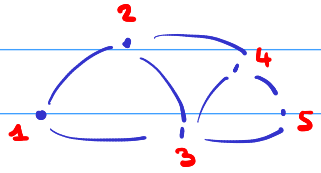
MODELLI 20/10/2026

Presentations: FRI 23/10 — 3 groups → Mattia, Chiara, Alessio / Alice, Olga, Lorenzo / Stefano, Nikolai, Paolo + Avio

≃ 20 min + discussion

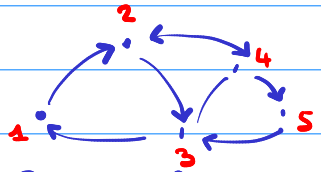
ROAD NETWORKS ~ (GRAPH MODELS)



Graph: $G = (V, E)$ — edges $\subseteq V \times V$

↳ vertices

UNDIRECTED GRAPH $V = \{1, \dots, 5\}$ $E = \{(1,2), (2,3) \dots\}$

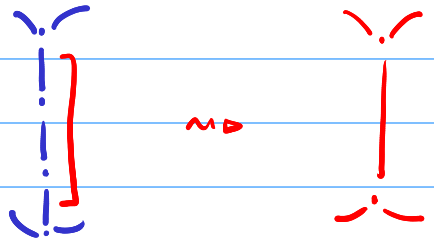


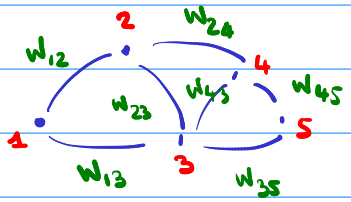
Assumptions: no self loops

DIRECTED GRAPH

ROAD NETWORKS: primal $V = \{\text{intersections}\}$ $E = \{\text{roads}\}$

dual $V = \{\text{roads}\}$ $i \leftrightarrow j$ iff they intersect





w_{ij} = "cost of going from i to j"

↳ WEIGHTED GRAPH

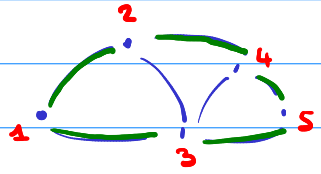
incidence matrix

n nodes (#V) m edges (#E)

$$A_{ij} = \begin{cases} 1 & \text{if } i \leftrightarrow j \\ 0 & \text{otherwise} \end{cases} \qquad A \in \mathbb{R}^{n \times n}$$

$$W_{ij} = \begin{cases} w_{ij} & \text{if } i \leftrightarrow j \\ 0 & \text{otherwise} \end{cases}$$ (weight matrix)

$d(i,j)$ = min length of walks / paths from $i \to j$



Ex: $d(1,4) = 2$

Unit distances: Find all shortest paths from i to all other nodes with breadth-first-search (BFS) (opposed to depth-first-search – DFS)

Positive weights: Dijkstra's algorithm solves the problem

⇝ Cost $O((m+n))$ BFS

$O((m+n)\log(m))$ Dijkstra

Properties of A: Path: seq. of edges

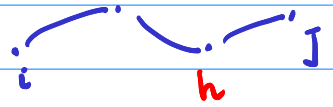
$(A^k)_{ij}$ = # { paths of length k connecting i to j }

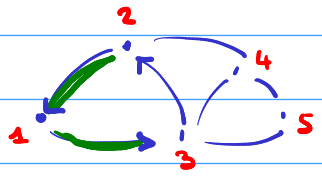
By induction: $k=0$ (identity) and $k=1$ are easy □

$k \Rightarrow k+1$ $\quad (A^{k+1})_{ij} = (A^k \cdot A)_{ij} = \sum_h (A^k)_{ih} A_{hj}$

$(A^k)_{ih}$: # paths of length k from i to h

A_{hj}: $=1$ if $h \leftrightarrow j$





d_i = "degree of node i" := # edges including i as an end point.

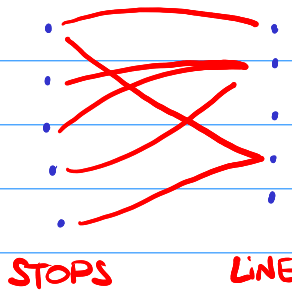
$(A^2)_{ii} = d_i \Rightarrow \operatorname{tr}(A^2) = 2m \quad m = \#E$

$(A^3)_{ii}$ = # triangles with i in it $\quad \operatorname{tr}(A^3) = 6 \cdot$ # triangles

Bipartite graphs

Ex: public transport Stops

Lines / Pub. service



Properties:

- no odd numbered cycles
- ...

$$A = \begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix} \begin{matrix} \text{stops} \\ \text{lines} \end{matrix} \qquad \Lambda(A) = \{\pm\lambda_i\}$$

Ex: $\begin{bmatrix} v \\ w \end{bmatrix}$ as candidate eigv.

$R = U\Sigma V^T$ ↳ unitary

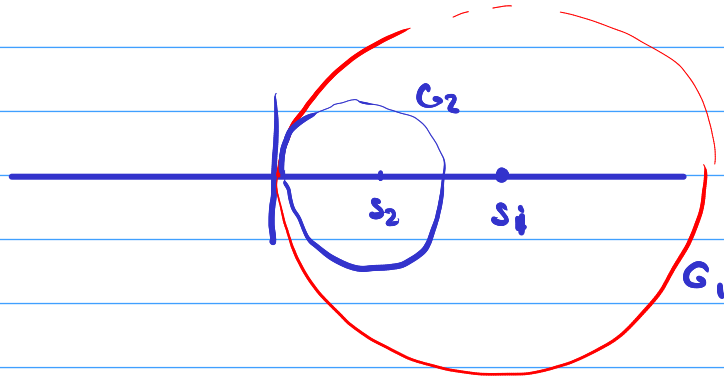
$\Sigma = \begin{bmatrix} \sigma_1 & & \sigma_n \\ & 0 & \end{bmatrix} \quad \sigma_i \geq 0$ ↳ singular values

$\lambda_i = \pm(\sigma_i(R))$

=

Graph Laplacian $\quad D_W = \begin{bmatrix} s_1 & & \\ & s_2 & \\ & & \ddots \\ & & & s_n \end{bmatrix} \quad s_i = \sum_j w_{ij}$

$L_W = D_W - W$ ~ Graph Laplacian $\quad \begin{bmatrix} \geq 0 & & \leq 0 \\ & \ddots & \\ \leq 0 & & = 0 \end{bmatrix}$



By Gerschgorin's thms

$\Lambda(L_W) \subseteq \cup C_i$

$\Rightarrow$ e.g. are non-negative

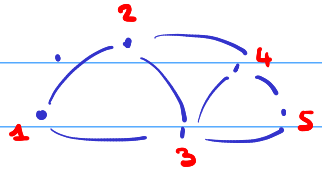
$\Rightarrow$ L_W is symmetric positive semidef.

Observation: $\mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \ker(L_W)$ (all rows sum to zero)

$B \in \mathbb{R}^{m \times n}$ ▷ auxiliary matrix

$e_k^T B = B(k,:) = [0 \dots 0 \overset{i}{-1} 0 \dots 0 \overset{j}{1} 0 \dots 0]$

edge $k = i \leftrightarrow j$



$B = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & 0 & 1 & -1 \end{bmatrix}$ $\begin{matrix} 1 \leftrightarrow 2 \\ 1 \leftrightarrow 3 \\ \\ \\ 4 \leftrightarrow 5 \end{matrix}$

$C = \begin{bmatrix} w_{e_1} & & \\ & \ddots & \\ & & w_{e_m} \end{bmatrix} \in \mathbb{R}^{m \times m}$

▷ weights of the edges

$\square = \square\square\square$

Claim $L_W = B^T C B$

Generic row of B: $e_i - e_j$ $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\{e_i - e_n\}$ canonical basis

$\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

$\overset{!}{=} \sum_{e=(i,j)} w_e (e_i - e_j)(e_i - e_j)^T = D_W - W$

$\begin{matrix} i \\ j \end{matrix} \begin{bmatrix} \ddots & & \\ & 1 & -1 \\ & -1 & 1 \\ & & \ddots \end{bmatrix}$

Rayleigh quotient : $R_{L_W}(x) = x^T L_W x \geq 0$

$= x^T B^T C B x \geq 0$

$l = 0 \iff Bx = 0$

Lemma Suppose G has q connected components $V_1, \dots, V_q$; then,

$\ker L_W = \text{span}\{\mathbb{1}_{V_1}, \dots, \mathbb{1}_{V_q}\}$

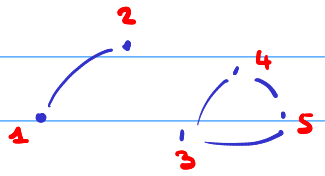
Proof $x \in \ker(L_W) \iff x^T L_W x = 0 \iff Bx = 0$

$\neq 0$

$Bx = 0$

$$\begin{bmatrix} | \\ x_i - x_j \\ | \end{bmatrix} \quad i \leftrightarrow j$$

This means $x_i = x_j$ for all $i \leftrightarrow j$



$x_1 = x_2$

$x_3 = x_4 = x_5$

$$\begin{bmatrix} \lambda \\ \lambda \\ \mu \\ \mu \\ \mu \end{bmatrix}$$

Without loss of gen.

$\lambda > 0$, $\mu < 0$,

if $\mathbb{1}^T x = 0$

For L_W $\lambda_1 = 0$ all the time (eigv $\mathbb{1}$)

$\lambda_2 = 0$ iff G is not connected

$$\lambda_2 = \min_{\substack{x \neq 0 \\ \mathbb{1}^T x = 0}} \frac{x^T L_W x}{x^T x}$$

$\Rightarrow$ Looking at signs in x, eigv to λ_2 orthogonal to $\mathbb{1}$, we can find members of each connected component.

