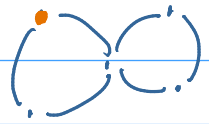
SUBGRAPH CENTRALITIES ⇝ similar to Katz



A_{ii} → # self-loops for i

A^2_{ii} → # walks of length 2 $i \to j \to i$

A^k_{ii} → # walks of length k $i \to \dots \to i$

$(A^k)_{ii}$

Q: How many closed walks of length k come back to i w.r.t to the maximum possible number?

↳ $k!$

$$C_{SUB}(i) = \sum_{k=0}^{+\infty} \frac{(A^k)_{ii}}{k!} = (e^A)_{ii} \qquad O(n^3) \text{ flops}$$

SIMILAR IDEA: Count all walks from i up to j for all possible $j \neq i$

↳ TOTAL COMMUNICABILITY

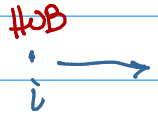
$$C_{TC}(i) = \sum_{k=0}^{+\infty} \frac{A^k \cdot \mathbb{1}}{k!} = e^A \cdot \mathbb{1} = e^A \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

⇝ Krylov methods for matrix functions

$$e^A \mathbb{1} = \left(I + A + \frac{A^2}{2} + \frac{A^3}{6} + \dots\right) \cdot \mathbb{1} = \mathbb{1} + A \cdot \mathbb{1} + \frac{A^2 \mathbb{1}}{2} + \dots$$

↳ since A is sparse ≈ $O(n)$

HITS ⇝ HUBS & AUTHORITIES ⇝ Natural for directed graphs



IDEA similar to eig. centrality

h = hub rankings

a = authority rankings

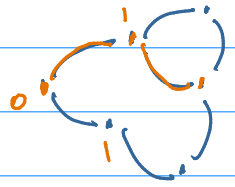
Good hubs subdivide their rank to the connected authorities
Good authorities are pointed to good hubs

$$a = \mu A h \qquad h = \mu A^T a \rightsquigarrow h = \mu^2 A^T A h$$

h, a are right and left singular vectors of A (or right eigv of A^TA and AA^T)

SVD $\rightsquigarrow O(n^3)$ — Natural choice: Power iteration for A^TA or AA^T.

PAGE RANK



$\rightsquigarrow$ Build a Markov chain — prob. of being in state i at time 0

$\pi_0^T = [(\pi_0)_1 \dots (\pi_0)_n]$

Time $\{0, 1, 2 \dots, \} = \mathbb{N}$

$X_t(i) = \mathbb{P}\{$ being at node i at time $t\}$

$\pi_0 = [1\ 0 \dots 0]$ → starting at node 1 for $t=0$

If we have d nodes connected to 1:

$\pi_1 = [0\ \frac{1}{d}\ 0\ \frac{1}{d} \dots \frac{1}{d}\ 0 \dots 0]$ — $\frac{1}{d}$ only at nodes connected with 1

$\pi_t^T = \pi_{t-1}^T P \qquad P = D^{-1}A$

↳ very similar to eigv. centrality

↳ incidence matrix

diag matrix with degrees

Easy to see: $\begin{cases} \exists \lim_{i \to +\infty} \pi_i \rightarrow \text{invariant measure} \\ \text{The limit does not depend on } \pi_0 \end{cases}$

Typically: ADD TELEPORTATION → pos. at time i = { with prob. γ: one of the conn. nodes; a random node }

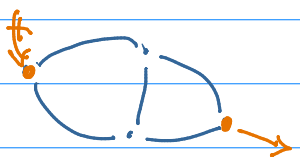
$\gamma \simeq 0{,}85$

with prob. $1-\gamma$

$$\hat{P} = P \cdot \gamma + (1-\gamma)\frac{\mathbb{1}\mathbb{1}^T}{n}$$

γ: random walk; $(1-\gamma)\frac{\mathbb{1}\mathbb{1}^T}{n}$: teleportation

RESISTIVE NETWORKS



Ohm's Law $\phi = RI = C^{-1}I$ $\quad C = 1/R$

Kirchoff's law $\sum I_i = 0$ at all nodes

Kirchoff $B^T I = \begin{bmatrix} \\ \\ \end{bmatrix}$ sum of all currents at a node

$B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ & & 1 & -1 \\ & & \vdots \end{bmatrix}$ (start, end)

$B^T I = b$ $\quad b_i = \begin{cases} 0 & \text{otherwise} \\ 1 & \text{at the entrance} \\ -1 & \text{at the exit} \end{cases}$

I: vector of currents in the edges

$\phi_i - \phi_j$ = diff of potential over the edge connecting $i \to j$

$C_e \cdot (\phi_i - \phi_j) = I_e \qquad e = i \to j$

$\Downarrow$

$CB\phi = I \Rightarrow \qquad B^T C B I = b = L_w I$

Can we solve $L_w \phi = I$? Singular, sol. ϕ is determined up to a multiple of $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

→ Solve with Moore-Penrose pseudoinverse $\phi = L_w^{\dagger} I$.