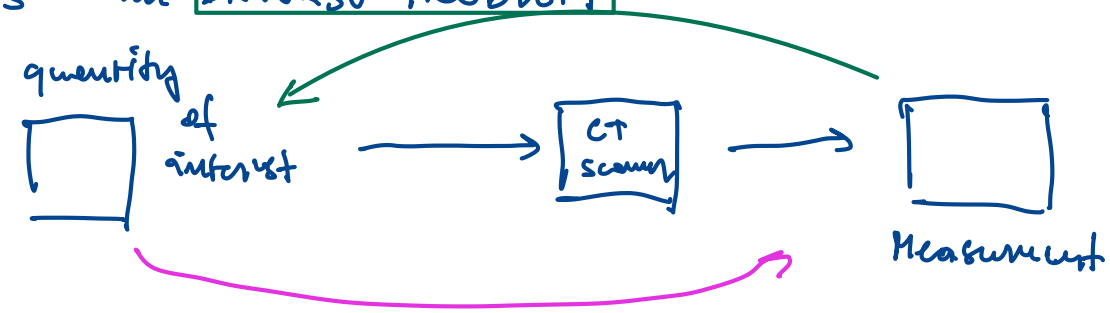
P. C. Hansen et. al. Computed Tomography. SIAM, 2021

(CT)

CT is an INVERSE PROBLEM (abstract / artificial)



FORWARD PROBLEM

(modeled after physical process)

Inverse Pb.s are ill-posed ;

Well-posed (Hadamard early XX cent.)

ill-posed = NOT (well-posed)

- (i) A solution exists
- & (ii) The solution is unique
- & (iii) The solution depends continuously on the data

CHALLENGING !

The PHYSICS behind CT (X-ray computed tomography)

/

Electromagnetic Radiations

(Frequency ↔ Wavelenght ↔ Energy ↔ Photons)

* POLYCHROMATIC (i.e., different energy levels)

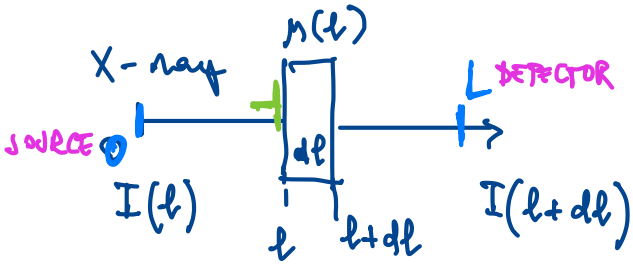
* MAIN PROPERTY: interact with an object / matter via (scattering & absorption)

(PHOTONS)

In practice, the X-rays are attenuated (in intensity) when they go through an object.

Under some simplifying assumptions, it is described by

LAMBERT-BEER LAW (ODE)



X-ray MONOCHROMATIC

$$dI \propto I(l)\cdot dl = -\mu(l)\, I(l)\, dl$$

$dI = I(l+dl) - I(l)$

$\mu(l)$: LINEAR ATTENUATION COEFFICIENT ≥ 0

$dl \to 0$

$$\begin{cases} \dfrac{dI}{dl} = -\mu(l)\, I(l) & \text{ODE} \\ I(0) = I_0 \end{cases}$$

With some derivation:

$$\int \frac{1}{I(l)}\, dI = \int -\mu(l)\, dl$$

$$\log\left(|I(l)|\right) + C = \int -\mu(l)\, dl$$

($|I(l)| \geq 0$; $\exp(\,)$ applied to both sides)

$$I(l)\exp(C) = \exp\left(-\int \mu(l)\, dl\right)$$

$$I(l) = \exp(-C)\exp\left(-\int \mu(l)\, dl\right)$$

$$I(0) = I_0 = \exp(-C)$$

$$I(l) = I_0 \exp\left(-\int \mu(l)\, dl\right)$$

RECORDED INTENSITY:

$$\boxed{I = I_0 \exp\left(-\int_0^L \mu(l)\, dl\right)}$$

Higher $\mu(l)$ $\rightsquigarrow$ lower I

Further manipulations:

$$I/I_0 = \exp\left(-\int_0^L \mu(l)\,dl\right) \rightsquigarrow -\log\left(\frac{I}{I_0}\right) = \int_0^L \mu(l)\,dl$$

(log, log)

TRANSMISSION

ABSORPTION (DATA)

IN the INVERSE PB. SETTING:

KNOWN

$\mu(l)$ is UNKNOWN

(i.e., the quantity we wish to describe)

This model is LINEAR w.r.t. $\mu(l)$

Generalization to POLYCHROMATIC X-rays:

$$I(E,l) = I_0(E)\exp\left(-\int \mu(E,l)\,dl\right) \quad \text{(More realistic setting)}$$

ENERGY

$$\text{DATA: } I(l) = \int_0^{E_{max}} I_0(E)\exp\left(-\int \mu(E,l)\,dl\right)dE$$

NO WAY THIS is LINEAR w.r.t. $\mu(E,l)$

From now on: STICK to the LINEAR MODEL

(Every kind of) DATA is affected by NOISE:

specifically, for tomography: detectors aquire I by counting photons

Once more, from Lambert-Beer LAW:

$$I = I_0 \cdot d \quad , \quad \underbrace{d = \exp\left(-\int_L \mu(l)\,dl\right)}_{\text{DECAY RATE}}$$

From a statistical viewpoint:

$$I \sim \mathcal{P}(I_0 d)$$

($\mathcal{P}$: POISSON)

Measurements are RANDOM VARIABLES

For $I_0 d$ large (for large intensities), thanks to CENTRAL LIMIT THM:

$$I \sim \mathcal{N}(I_0 d, I_0 d)$$

equivalently:

$$\boxed{I = I_0 d + \sqrt{I_0 d}\, Z}, \quad Z \sim \mathcal{N}(0,1)$$

$I_0 d$: Lambert-Beer LAW

$\sqrt{I_0 d}\, Z$: Gaussian Noise

$Z \sim \mathcal{N}(0,1)$: Std Normal-distributed Random Variable

THE MATHS behind CT: RADON TRANSFORM / INVERSE RADON TRANSF.

RADON TRANSFORM $\simeq$ forward modelling

INVERSE RADON TRANSF. $\simeq$ inverse pb.

Set a parametrization for LINES in $\mathbb{R}^2$

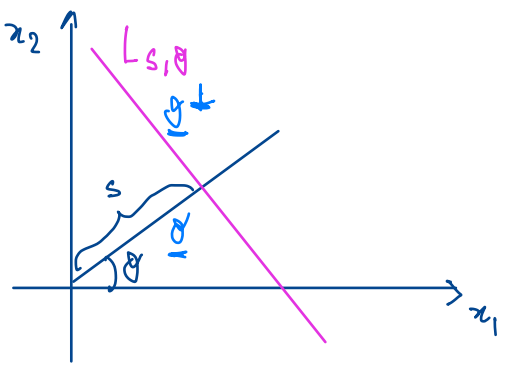
Take $\underline{\theta} = [\cos(\theta), \sin(\theta)]^T$, $\underline{\theta}^\perp = [-\sin(\theta), \cos(\theta)]^T$

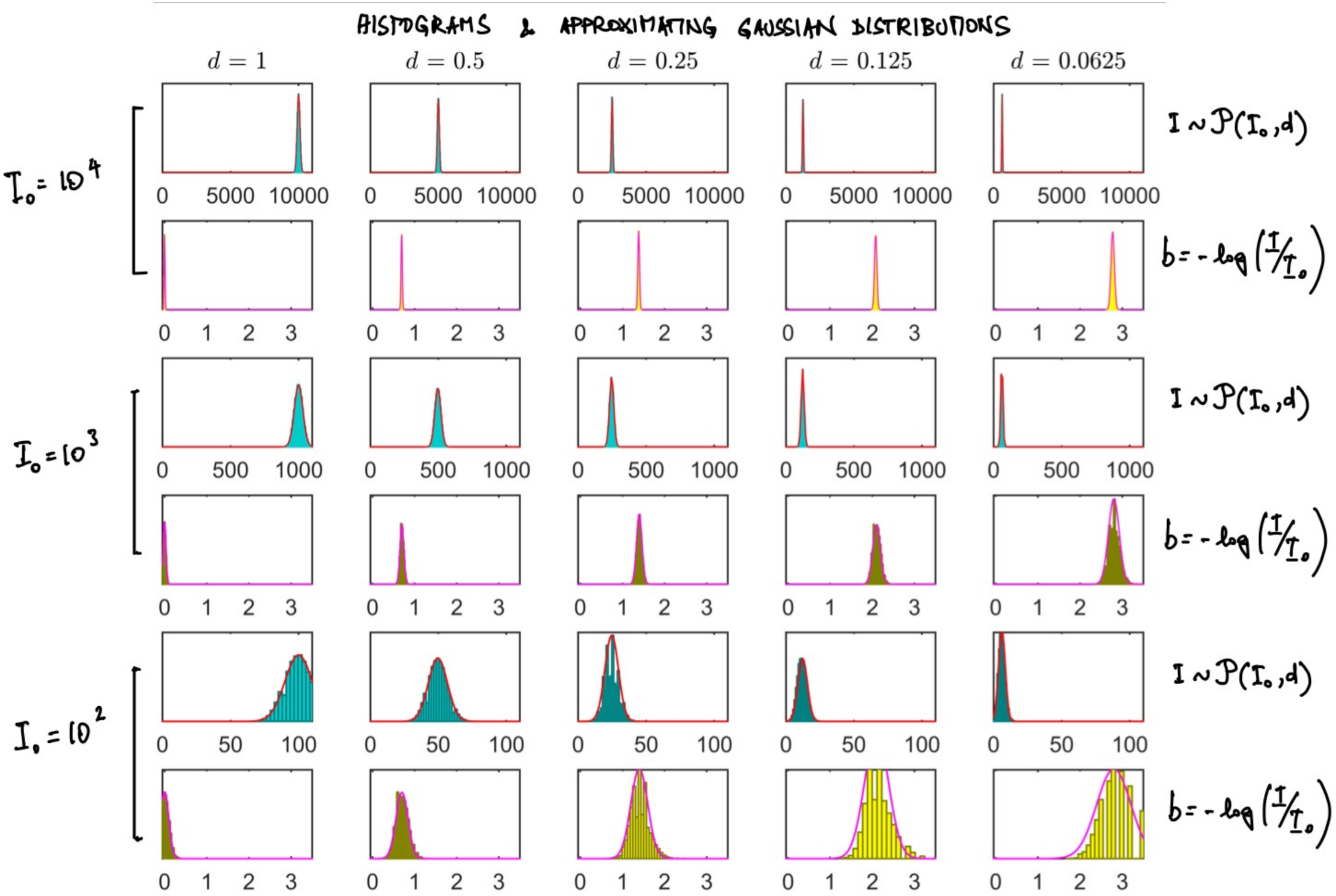
(θ: angle)

Given $s \in \mathbb{R}$, the set

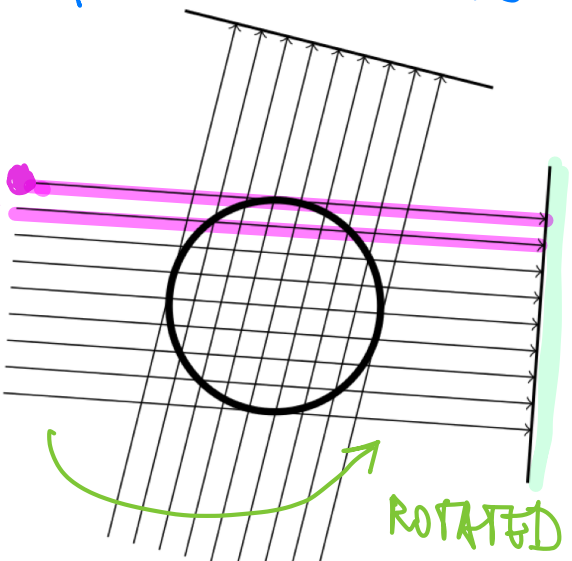
$$L_{s,\theta} = \left\{ \underline{x} = [x_1, x_2]^2 \in \mathbb{R}^2 \;\middle|\; \underline{x} \cdot \underline{\theta} = s \right\}$$

($\cdot$: inner prod.)




HISTOGRAMS & APPROXIMATING GAUSSIAN DISTRIBUTIONS
$d = 1$
$d = 0.5$
$d = 0.25$
$d = 0.125$
$d = 0.0625$
$I_0 = 10^4$
$I_0 = 10^3$
$I_0 = 10^2$
$I \sim \mathcal{P}(I_0, d)$
$b = -\log\left(\frac{I}{I_0}\right)$
0 5000 10000
0 500 1000
0 50 100
0 1 2 3

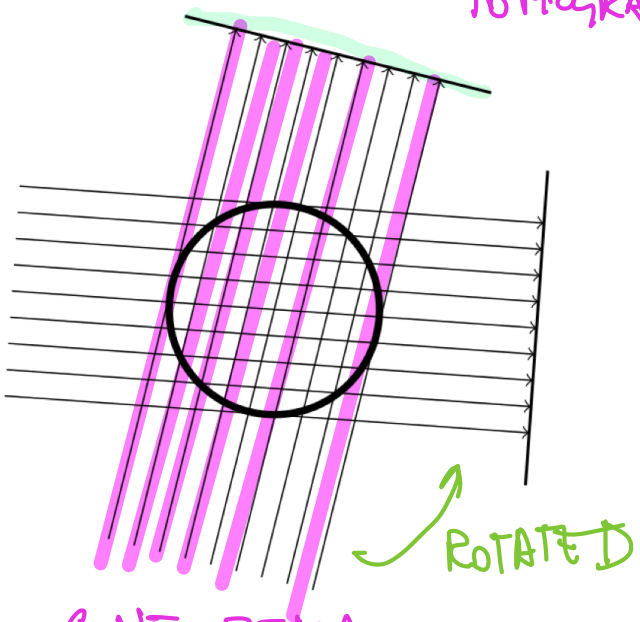
ACQUISITION GEOMETRIES



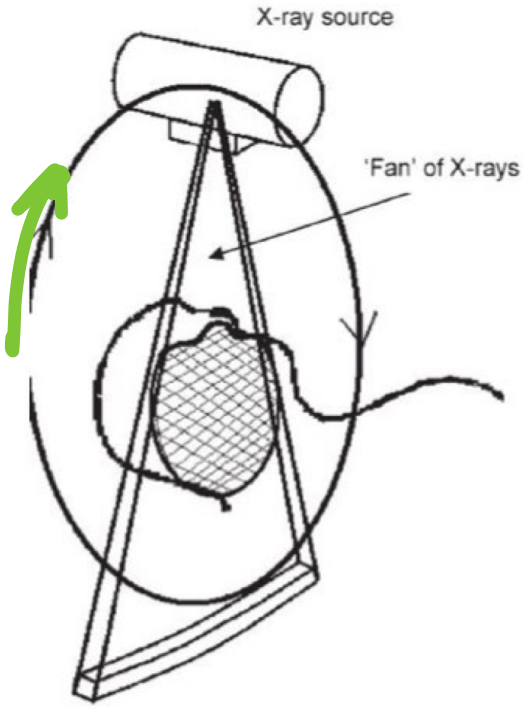
"pencil" set-up
(sequentially)

Match LAMBERT-BEER

PARALLEL-BEAM
TOMOGRAPHY



FAN-BEAM



CONE-BEAM for 3D IMAGING

