The story so far:

① Computed Tomography (CT) as an INVERSE PB.:

GOAL: recover the (spatially variant) LINEAR ATTENUATION COEFFICIENT $\mu(\ell)$ (i.e., describe the interior of an object) from ABSORPTION DATA d (i.e., measurements of the decay of the X-ray intensity)

② FORWARD PB. (Physical Model underlying the measurement process)

LAMBERT-BEER LAW:

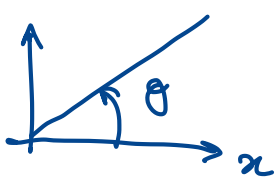
$$I = I_0 \exp\left(-\int_L \mu(\ell)\,d\ell\right) \iff d = -\log\left(\tfrac{I}{I_0}\right) = \int_L \mu(\ell)\,d\ell$$

(LINEAR in $\mu(\ell)$)

Notes:
- simplifying assumptions
- d typically affected by (Gaussian) noise

Radon Transform as a mathematical tool to express the forward pb.

Given ϑ



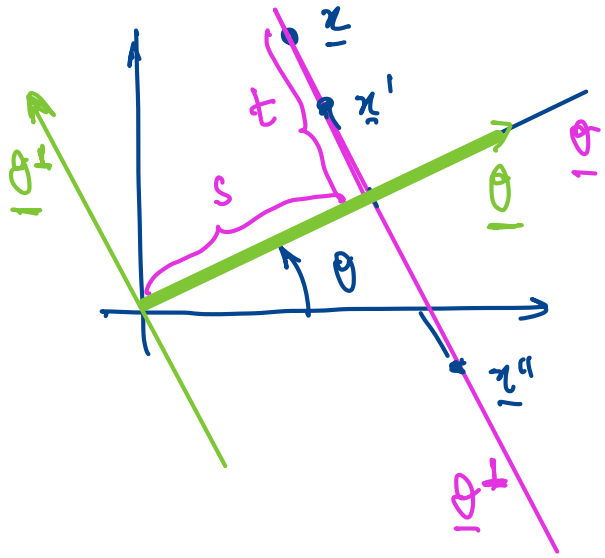
,

$$L_{s,\vartheta} = \left\{ \underline{x} \in \mathbb{R}^2 \;\middle|\; \underline{x} \cdot \underline{\vartheta} = s \right\}$$

$\underline{x} = (x, y)$; $\cdot$: INNER PRODUCT

IN NORMAL FORM

$$\underline{\vartheta} = \begin{bmatrix} \cos(\vartheta) \\ \sin(\vartheta) \end{bmatrix}, \quad \underline{\vartheta}^{\perp} = \begin{bmatrix} -\sin(\vartheta) \\ \cos(\vartheta) \end{bmatrix}$$



In particular : $\|\underline{\theta}\|_2 = \|\underline{\theta}^\perp\|_2 = 1$, $\underline{\theta} \cdot \underline{\theta}^\perp = 0$

Change of basis :

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{=: P_\theta \text{ ROTATION MATRIX}} \underbrace{\begin{bmatrix} s \\ t \end{bmatrix}}_{\underline{s}}$$

$$\underline{s} = P_\theta^T \underline{x}$$

$$= s\,\underline{\theta} + t\,\underline{\theta}^\perp$$

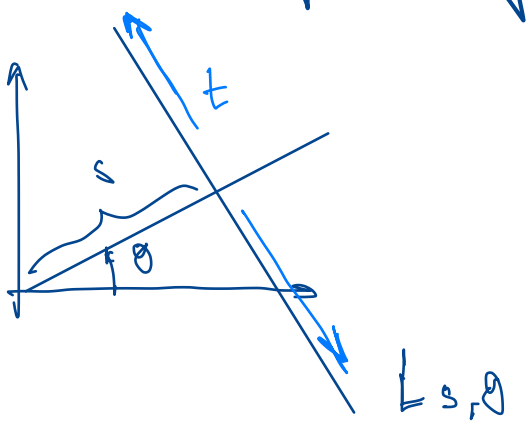
We define : RADON TRANSFORM of f

$$\mathcal{R}f(s,\theta) = \int_{L_{s,\theta}} f(x,y)\, dl \qquad \forall s \in \mathbb{R},\ \forall \theta \in [0,\pi)$$

with a change of variable :

$$x(t) = \cos(\theta)\, s - \sin(\theta)\, t$$
$$y(t) = \sin(\theta)\, s + \cos(\theta)\, t$$

I am integrating along $L_{s,\theta}$: s, θ fixed; $t \in \mathbb{R}$ variable



So:

$$Rf(s,\theta) = \int_{-\infty}^{+\infty} f\Big(\cos(\theta)s - \sin(\theta)t,\ \sin(\theta)s + \cos(\theta)t\Big)\cdot 1 \cdot dt$$

$$= \int_{-\infty}^{+\infty} f\left(s\underline{\theta} + t\underline{\theta}^{\perp}\right) dt$$

CONNECTS to the LAMBERT-BEER LAW by TAKING:

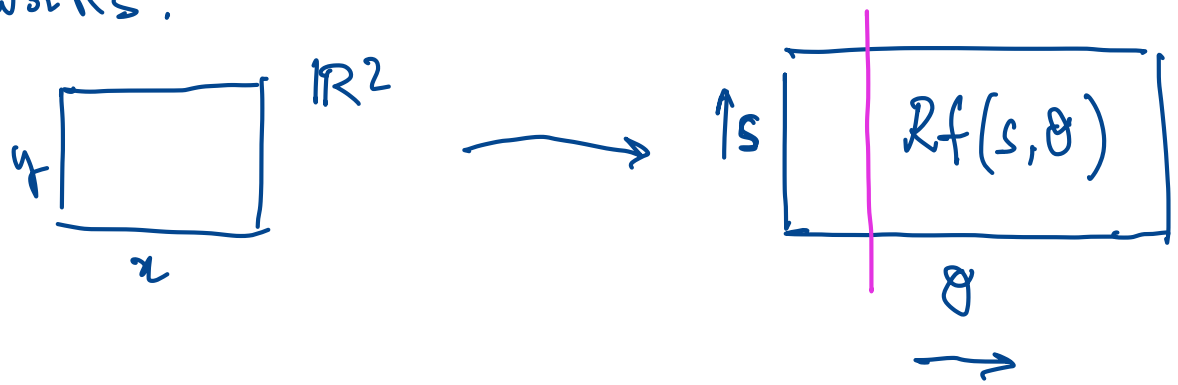
$$f(x,y) = \underbrace{\mu(x,y)}_{\text{ATTENUATION COEFFICIENT}}$$

$$\overbrace{\quad\quad}^{\text{ABSORPTION}} = -\log\left(\frac{I_{s,\theta}}{I_0}\right) = \int_{L_{s,\theta}} \mu(x,y)\,dl$$

is the RADON TRANSFORM of

Radon theory: ways to "invert the Radon transf." i.e., recover f (i.e. μ) given Rf (i.e. d)

Some more about how the Radon transform works:



the output of the Radon transform (for varying s and ϑ) is the SO-CALLED

SINOGRAM

vertical lines in the sinogram correspond to the data detected by sensors paired with parallel beams at a single angle ($\leftrightarrow$ PARALLEL BEAM ACQUISITION GEOMETRY)

Steps for INVERTING the RADON TRANSFORM:

TOOLS: $\mathcal{F}$ FOURIER TRANSFORM

$\mathcal{B}$ BACK-PROJECTION OPERATOR

$$\mathcal{B}\, g\,(x,y) = \frac{1}{\pi}\int_0^{\pi} g\left(\underbrace{x\cos(\vartheta)+y\sin(\vartheta)}_{\text{off-set } s},\ \vartheta\right) d\vartheta$$

(x,y): cartesian coordinates

g: Radon-transform data

off-set s corresponding to a line $L_{s,\vartheta}$, that

$\mathcal{B}$ should "REVERT" of $\mathcal{R}$ (think of $g = \mathcal{R}f$)

goes through (x, y) (check the change of variables ✱)

$$f \longrightarrow \mathcal{R}f \longrightarrow \mathcal{B}\overline{\mathcal{R}f}^{\,=g}$$

INTEGRAL INTEGRAL

just averaging: $f \neq \mathcal{B}\mathcal{R}f$

$\mathcal{B} \neq \mathcal{R}^{-1}$; can be proven that

$$\langle g, \mathcal{R}f \rangle = \langle \mathcal{B}g, f \rangle$$

ADJOINT

We need: FILTERED BACK-PROJECTION (FBP) FORMULA

(proof. based: CENTRAL SLICE THEOREM)

$$f(x,y) = \frac{1}{2}\,\mathcal{B}\left(\mathcal{F}^{-1}\left(|\omega|\,\mathcal{F}\left(\underbrace{\mathcal{R}f}_{\text{DATA}}\right)(\omega,\vartheta)\right)\right)(x,y)$$

$f(x,y) \longrightarrow \mathcal{R}f(s,\vartheta) \xrightarrow[\substack{\text{FOURIER}\\ \text{TRANSF.}}]{\text{1D}} \mathcal{F}(\mathcal{R}f) \xrightarrow{|\omega|}$ "FILTER"

$(s \rightsquigarrow \omega)$ FREQUENCY

FREQUENCY DOMAIN

is ESSENTIAL! $|\omega| = 1 \rightsquigarrow \frac{1}{2}\mathcal{B}(\underbrace{\mathcal{F}^{-1}\mathcal{F}}_{=\mathrm{id}}\mathcal{R}f)$

FBP is a formula for inversion, but, in theory you need $Rf(s,\theta)$, $\forall s, \forall \theta$

IN PRACTICE: NO continuous, full acquisition (Rf)

Moreover: FBP is sensitive to the noise in Rf (i.e., the data)

which is AMPLIFIED in f

Intuition: Rf acquired with NOISE

very oscillating, high frequency component in Rf

$f \rightarrow Rf \rightarrow \mathcal{F}Rf$

w.r.t. the variable ω, frequency

because of noise, $\mathcal{F}Rf \neq 0$ for $|\omega|$ LARGE

$\rightarrow |\omega| \mathcal{F}Rf$ is EVEN LARGER than $\mathcal{F}Rf$ for $|\omega|$ LARGE

$\rightarrow$ keep applying the inversion formula

MAKES the INVERSION ILL-POSED !

WAY OUT: REPLACE $|\omega|$ by LOW-PASS FILTER (allows to stabilize the reconstruction)

This is a FORM of REGULARIZATION !