What is this course about

Not approximation methods. (!)

A few selected advanced topics in linear algebra, close to (some of) the themes of our research group in Pisa.

New teacher, Federico Poloni (Inst. of Computer Science); partially new topics.

Themes

- Matrix pencils and polynomials (canonical forms / structure);
- Methods to compute for matrix functions;
- (Some) methods to solve matrix equations.

Course features

Prereqs

- Numerical analysis
- Scientific computing

Synergizes with other courses from the same area.

Course format

- Frontal lectures — will attempt to record them.
- Seminars by the students: last year 2/each (one during the course, one after). We'll see, also depending on no. of students.

Material from several books / sources.

Possibly some changes along the way — new course for me, too.

Movie: matrix pencils

Generalized eigenvalue problems: $\det(A - \lambda E) = 0$, $E \neq I$.

Some new features; for instance, eigenvalues at ∞

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Some more pathological cases; for instance,

$$\det \begin{bmatrix} 0 & 1 & \lambda \\ 1 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix} \equiv 0.$$

Try conjugating and computing its eigenvalues with `eig(Q*A*Q', Q*E*Q')`...

What can we say about higher-degree matrix polynomials?

Movie trailer: matrix functions

How to define $f(A)$ for an analytic function f? You have already seen $\exp(A)$...
Either via a series expansion or

$$f(A) = f(V\Lambda V^{-1}) = V \operatorname{diag}(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_m)) V^{-1}.$$

Higher derivatives may pop up unexpectedly:

$$f\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} f(0) & f'(0) & f''(0) \\ 0 & f(0) & f'(0) \\ 0 & 0 & f(0) \end{bmatrix}.$$

Techniques to compute them involve Cauchy integrals, interpolation...

Movie trailer: matrix equations

Algebraic Riccati equations

Find $X \in \mathbb{R}^{n \times n}$ that solves

$$XCX - AX + XD - B = 0.$$

Appears in several applications, e.g., control theory.

(Block) eigenvalue problem in disguise: find X, $\Lambda = CX + D$ s.t.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ I \end{bmatrix} = \begin{bmatrix} X \\ I \end{bmatrix} \Lambda.$$

Or: find X such that

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

Movie trailer: matrix sign

Newton for the matrix sign

$A_{k+1} = \frac{1}{2}(A_k + A_k^{-1}), \qquad A_0 = A.$

Maps eigenvalues according to $\lambda_i^{(k+1)} = \frac{1}{2}(\lambda_i^{(k)} + 1/\lambda_i^{(k)})$.
Two limit fixed points, ± 1.

Converges to the matrix analogue of the sign function,

$$\operatorname{sgn}(A) = \operatorname{sgn}(VDV^{-1}) = V(\operatorname{sgn}(\lambda_1), \operatorname{sgn}(\lambda_2), \ldots, \operatorname{sgn}(\lambda_m))V^{-1}.$$

Splits the spectrum of A in two: $\ker(A_\infty - I)$ and $\ker(A_\infty + I)$.

- Can be used to solve algebraic Riccati equations.
- Can be used to find eigenvalues recursively, as a "matrix product-heavy" algorithm.