Matrix pencils

Definition: Matrix pencil

$A + xB$, with $A, B \in \mathbb{C}^{m\times n}$, x indeterminate.

A pencil is called regular if $n = m$ and $\det(A + xB)$ does not vanish identically, i.e., if there is $\lambda \in \mathbb{C}$ for which it is square invertible.

An eigenvalue λ is a value for which $\det(A + \lambda B) = 0$.
Eigenvector, Jordan chains...

If $\det(A + xB)$ has degree less than n, the 'missing' eigenvalues are said to be "at infinity".

Example

$$\begin{bmatrix} x+1 & x \\ x & x+1 \end{bmatrix} \quad \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

Eigenvalues of singular pencils

Can be defined via ‘unusual rank drop’. For instance:

$$A + xB = \begin{bmatrix} 2 & x & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ x & x & 0 \end{bmatrix}$$

has typical rank 2. More formally, $\operatorname{rank}_{\mathbb{C}(x)}(A + xB) = 2$.
But $A + 2B$ has rank 1.

Canonical form

Equivalence relation $\sim$: for each two square $P \in \mathbb{C}^{m \times m}, Q \in \mathbb{C}^{n \times n}$ square invertible, $A + xB$ and $P(A + xB)Q$ are said to be equivalent.

Equivalent $\implies$ same eigenvalues, singularity…

If B is square nonsingular, there is little new in this theory: $A + xB \sim J - xI$, where J is the Jordan canonical form of $-B^{-1}A$ (or $-AB^{-1}$).
Computing eigenvalues of $A + xB \iff$ computing eigenvalues of $-B^{-1}A$

Theorem (Weierstrass canonical form)

For a regular matrix pencil $A + xB \in \mathbb{C}[x]^{n\times n}$, there are nonsingular $P, Q \in \mathbb{C}^{n\times n}$ such that $P(A + xB)Q$ is the direct sum (`blkdiag`) of blocks of the forms

$$J_\lambda(x) = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} - xI, \quad J_\infty(x) = I - x \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Proof (sketch):

- Take c such that $A + cB$ is invertible;
- $A + xB \sim I + (x - c)(A + cB)^{-1}B$;
- $A + xB \sim I + (x - c)\,\mathrm{blkdiag}(J_1, \dots, J_s)$,
- Consider separately each $I + (x - c)J_i = I + (x - c)(\lambda I + N)$.
- If $\lambda = 0$, block $\sim I - xM$, where $M = \mathrm{toeplitztriu}(0, 1, \dots)$.
- If $\lambda \neq 0$, block $\sim M - xI$, where $M = \mathrm{toeplitztriu}(\frac{c\lambda - 1}{\lambda}, \frac{1}{\lambda^2}, \dots)$.

One can define Jordan chains:

at λ: $-Av_0 = \lambda Bv_0$, $-Av_1 = \lambda Bv_1 + Bv_0, \ldots$

at ∞: $-Bv_0 = 0$, $-Bv_1 = Av_0, \ldots$

Generalized Schur factorization

Compare with generalized Schur (QZ) factorization:

Theorem

For any pair of square $A, B \in \mathbb{C}^{m \times m}$, one can find orthogonal Q, Z such that $QAZ = T_A, QBZ = T_B$ are upper triangular (at the same time).

Eigenvalues $= \frac{(T_A)_{ii}}{(T_B)_{ii}}$ (incl. ∞).

Theorem (Kronecker canonical form)

For a ~~regular~~ matrix pencil $A + xB \in \mathbb{C}[x]^{m\times n}$, there are nonsingular $P \in \mathbb{C}^{m\times m}, Q \in \mathbb{C}^{n\times n}$ such that $P(A + xB)Q$ is the direct sum (`blkdiag`) of blocks of the form $J_\lambda(x), J_\infty(x)$, and

$$\begin{bmatrix} 1 & x & & & \\ & 1 & x & & \\ & & \ddots & \ddots & \\ & & & 1 & x \end{bmatrix} \in \mathbb{C}[x]^{k\times(k+1)}, \quad \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & x & \ddots & \\ & & \ddots & 1 \\ & & & x \end{bmatrix} \in \mathbb{C}[x]^{(k+1)\times k},$$

(This includes 1×0 and 0×1 empty blocks).

Examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & x \\ 1 & x \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & x \\ 1 & 0 & 0 \\ x & 0 & 0 \end{bmatrix} \ldots$$

Proof (sketch): [Gantmacher book '59]

- Suppose $(A + xB)v(x) = 0$ for some $v \in \mathbb{C}(x)^n$
- We may assume $v = v_0 + v_1 x + \cdots + v_d x^d \in \mathbb{C}[x]^n$, clearing denominators.
- Remark: singularity of $(d+1) \times d$ $\begin{bmatrix} A & & & \\ B & A & & \\ & \ddots & \ddots & \\ & & B & A \\ & & & B \end{bmatrix}$.
- Assume d minimal.
- We wish to show that the v_i are linearly independent. Suppose they are not so; then one can choose $\alpha(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_e x^e$ (of minimal degree $e \leq d$) such that $w(x) = \alpha(x)v(x)$ has a zero coefficient w_e. But then $Aw_0 = 0$, $Aw_1 + Bw_0 = 0$, ..., $Bw_{e-1} = 0$, which contradicts minimality of d.

(cont.)

- Take a basis that starts with the v_i; this block-triangularizes the pencil: $\begin{bmatrix} K(x) & L(x) \\ 0 & M(x) \end{bmatrix}$, where $K(x)$ is a Kronecker block.
- By the minimality of d, $M(x)$ is such that $d \times (d-1)$

$$\begin{bmatrix} M_0 & & & \\ M_1 & M_0 & & \\ & \ddots & \ddots & \\ & & M_1 & M_0 \\ & & & M_1 \end{bmatrix}$$ is nonsingular.
- Using this nonsingularity, one can prove that the system of Sylvester-like equations

$$\begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \begin{bmatrix} K(x) & L(x) \\ 0 & M(x) \end{bmatrix} \begin{bmatrix} I & F \\ 0 & I \end{bmatrix} = \begin{bmatrix} K(x) & 0 \\ 0 & M(x) \end{bmatrix}$$

is solvable (some work needed — details not given).

Kernel in $\mathbb{C}(x)$

The $(k \times (k+1))$ Kronecker blocks have kernel
$\begin{bmatrix}(-1)^k x^k & (-1)^{k-1}x^{k-1} & \dots & -x & 1\end{bmatrix}^T$.
The other blocks have full column rank in $\mathbb{C}(x)$.

This can be used to characterize $\ker_{\mathbb{C}(x)}(A + Bx)$, using the fact that $\ker \operatorname{blkdiag}(C, D) = \operatorname{blkdiag}(\ker C, \ker D)$.

Remark: this gives a minimal basis, i.e., all other polynomial bases of $\ker_{\mathbb{C}(x)}(A + Bx)$ have higher degrees.