Polynomials of matrices

Another different way to make polynomials and matrices interact: take a scalar polynomial, and apply a (square) matrix to it, e.g.,

$$p(x) = 1 + 3x - 5x^2 \implies p(A) = I + 3A - 5A^2.$$

Lemma

If $A = S\,\mathrm{blkdiag}(J_1, J_2, \dots, J_s)S^{-1}$ is a Jordan form, then $p(A) = S\,\mathrm{blkdiag}(p(J_1), p(J_2), \dots, p(J_s))S^{-1}$, and

$$p(J_i) = \begin{bmatrix} p(\lambda_i) & p'(\lambda_i) & \dots & \frac{1}{k!}p^{(k)}(\lambda_i) \\ & p(\lambda_i) & \ddots & \vdots \\ & & \ddots & p'(\lambda_i) \\ & & & p(\lambda_i) \end{bmatrix}.$$

(Proof: Taylor expansion of p around λ.)

Valutare polinomio scalare in un blocco di Jordan:

$J = \lambda I + N \qquad N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$

$$p(x) = p(\lambda) + p'(\lambda)(x-\lambda) + \dots + \frac{1}{d!} p^{(d)}(\lambda)(x-\lambda)^d$$

$$p(J) = p(\lambda) I + p'(\lambda)(J - \lambda I) + \dots + \frac{1}{d!} p^{(d)}(\lambda)(J - \lambda I)^d =$$

$$= p(\lambda) I + p'(\lambda) N + \frac{1}{2} p''(\lambda) N^2 + \dots + \frac{1}{d!} p^{(d)} N^d =$$

$$= \begin{bmatrix} p(\lambda) & p'(\lambda) & & \cdots \\ & \ddots & \ddots & \\ & & \ddots & p'(\lambda) \\ & & & p(\lambda) \end{bmatrix}$$

Functions of matrices [Higham book, '08]

We can extend the same definition to arbitrary scalar functions:

Given a function $f : U \subseteq \mathbb{C} \to \mathbb{C}$, we say that f is defined on A if f is defined and differentiable at least $\underline{m_g(\lambda_i) - 1}$ times on each eigenvalue λ_i of A.

Definition

If $A = \underline{S}\,\mathrm{blkdiag}(J_1, J_2, \dots, J_s)\underline{S}^{-1}$ is a Jordan form, then $f(A) = \underline{S}\,\mathrm{blkdiag}(f(J_1), f(J_2), \dots, f(J_s))\underline{S}^{-1}$, where

$$\boxed{f(J_i) = \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \dots & \frac{1}{k!}f^{(k)}(\lambda_i) \\ & f(\lambda_i) & \ddots & \vdots \\ & & \ddots & f'(\lambda_i) \\ & & & f(\lambda_i) \end{bmatrix}.}$$

(Reasonable doubt: is it independent of the choice of S?)

Def: Hermite interpolating polynomial:

Dati nodi $\lambda_1, \lambda_2, \dots \lambda_k$, molteplicità $1 \leq m_i$ per $i = 1, 2, \dots k$, e una funzione derivabile $m_i - 1$ volte in λ_i, esiste un (unico) polinomio di grado $\left(\sum_i m_i\right) - 1$ tale che $p^{(j)}(\lambda_i) = f^{(j)}(\lambda_i)$ per ogni $i = 1, 2, \dots k$, $j = 1, 2, \dots m_i$

ES: trovare un polinomio tale che

$p(0) = 1 \qquad p(1) = e$

$\to \; p'(0) = 1 \qquad p'(1) = e$

$\to \; p''(0) = 1 \qquad p''(1) = e$

(coincide con $\exp(x)$ in 0 e 1, con due derivate ognuno)

$(\deg = 5)$

$$
\begin{array}{c|cccc}
x_1 & f(x_1) & f[x_1,x_2] & f[x_1,x_2,x_3] & f[x_1,x_2,x_3,x_4] \\
x_2 & f(x_2) & f[x_2,x_3] & f[x_2,x_3,x_4] & \\
x_3 & f(x_3) & f[x_3,x_4] & & \\
x_4 & f(x_4) & & &
\end{array}
$$

(forma di Newton)

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

$\vdots$

Alternate definition: via Hermite interpolation

Definition

$f(A) = p(A)$, where p is a polynomial such that $f(\lambda_i) = p(\lambda_i), f'(\lambda_i) = p'(\lambda_i), \dots, f^{(m_g(\lambda_i)-1)}(\lambda_i) = p^{(m_g(\lambda_i)-1)}(\lambda_i)$ for each i.

We may use this as a definition of $f(A)$ (and it does not depend on S).

Obvious from the definitions that it coincides with the previous one.

Remark: if $A \in \mathbb{C}^{m \times m}$ has multiple Jordan blocks with the same eigenvalue, these may be fewer than m conditions.
Remark: be careful when you say "all matrix functions are polynomials", because p depends on A.

Se f è un polinomio, allora $f(A)$ definita in questo modo coincide con il valutare $f(x)$ in A:

$A = SJS^{-1}$

valutare $f(x)$ in A produce $S \cdot \mathrm{blkdiag}(f(J_1), \dots f(J_k)) S^{-1}$

dove $f(J_i) = \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{1}{(m_i-1)!} f^{(m_i-1)}(\lambda_i) \\ & \ddots & \ddots & \vdots \\ & & \ddots & f(\lambda_i) \\ & & & f(\lambda_i) \end{bmatrix}$

che coincide con $p(J_i)$ se $p(i)$ è il polinomio di interpolazione.

Quindi $f($

Some properties

- If the eigenvalues of A are $\lambda_1, \dots, \lambda_s$, the eigenvalues of $f(A)$ are $f(\lambda_1), \dots, f(\lambda_s)$. (Remark: geometric multiplicities may drop)
- $f(A)g(A) = g(A)f(A) = (fg)(A)$ (since they are all polynomials in A).
- If $f_n \to f$ together with 'enough derivatives' (for instance because they are analytic and the convergence is uniform), then $f_n(A) \to f(A)$.
- If a sequence of matrix $A_n \to A$, then $f(A_n) \to f(A)$.
  Proof: let p_n be the (Hermite) interpolating polynomial on the eigenvalues of A_n. Interpolating polynomials are continuous in the nodes, so $p_n \to p$ (coefficient by coefficient). Then
  $\|p_n(A_n) - p(A)\| \leq \|p_n(A_n) - p_n(A)\| + \|p_n(A) - p(A)\| \leq \dots$.

Examples

- $\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$. The identity holds also for non-diagonalizable matrices (by continuity).
- $\operatorname{sgn}(A)$ from

$$\operatorname{sgn}(x) = \begin{cases} -1 & \operatorname{Re} x < 0, \\ 1 & \operatorname{Re} x > 0, \\ \text{undefined} & \operatorname{Re} x = 0. \end{cases}$$

- $A^{1/2}$ from $f(x) = \sqrt{x}$. Note that we can choose signs (branch) independently on each eigenvalue. All the various ways satisfy $(A^{1/2})^2 = A$.

Nonprimary matrix functions

If a matrix A has multiple eigenvalues, one could also define a 'square root' by choosing different signs on Jordan blocks with the same eigenvalue: for instance, $\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$ as a square root of I_2 (or also $V \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} V^{-1}$ for any invertible $V \dots$).

These are called nonprimary matrix functions (and they are not matrix functions with our definition).

(They all satisfy $f(A)^2 = A$.)

(They are not polynomials in A).

Cauchy integrals

If f is analytic on and inside a contour Γ that encloses the eigenvalues of A,

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)(zI - A)^{-1}dz.$$

Generalizes the analogous scalar formula.

Proof If $A = V\Lambda V^{-1} \in \mathbb{C}^{m\times m}$ is diagonalizable, the integral equals

$$V\begin{bmatrix} \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\lambda_1}dz & & \\ & \ddots & \\ & & \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\lambda_m}dz \end{bmatrix} V^{-1} = V\begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_m) \end{bmatrix} V^{-1}.$$

By continuity, the equality holds also for non-diagonalizable A.

Methods

Matrix functions arise in several areas: solving ODEs (e.g. $\exp A$), matrix analysis (square roots), physics, ...

Main methods to compute them:

- Factorizations (eigendecompositions, Schur...),
- Matrix versions of scalar iterations (e.g., Newton on $x^2 = a$),
- Interpolation / approximation,
- Complex integrals.