

# Lyapunov equations

## Lyapunov equation

$$\boxed{A^*X + XA + Q = 0}, \quad Q = Q^* \succeq 0. \quad (L)$$

*pos. semi-def. uita*

Special case of the Sylvester equation.

## Lemma

Suppose (L) has a unique solution  $X$ ; then  $X$  is symmetric.

**Proof:** transpose everything;  $X^*$  is another solution.

$$\underline{(L)^* : X^*A + A^*X^* + Q = 0}$$

(Come risolveremo  $AX - XB = C$ ? Prendiamo  
 cambi di base  $A = Q_A^T A Q_A^*$   $B = Q_B^T B Q_B^*$ ,

$$Q_A^* Q_A^T A Q_A^* X Q_B - Q_A^* X Q_B^T B Q_B^* Q_B = Q_A^* C Q_B$$

$$\Rightarrow T_A \hat{X} - \hat{X} T_B = \hat{C}, \quad \hat{X} = Q_A^* X Q_B, \quad \hat{C} = Q_A^* C Q_B$$

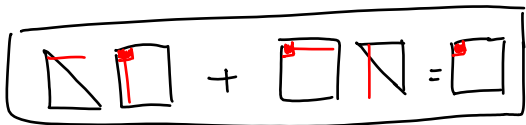
che risolviamo per sostituzione, un elemento di  $\hat{X}$   
 per volta e partite da  $\hat{X}_{n,1}$ :

$$\begin{array}{|c|} \hline \diagdown \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} - \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \\ \hline \end{array}$$

Nel caso Lyapunov, basta una forma di

Schur:  $A = Q_A T_A Q_A^*$ , che produce

$$T_A^* \hat{X} + \hat{X} T_A = \hat{Q}$$



serve che  $\lambda_i + \lambda_j^* \neq 0$   
 $\forall i, j$ , dove  $\lambda_i$  autovel. di  $A$

che posso risolvere per sostituzione a partire  
da  $\hat{X}_{1,1}$ .

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Altra osservazione: (L) ha solut. unica se e  
solo se  $\lambda_i + \lambda_j^* \neq 0 \forall i, j$  ( $\lambda_i$  autovel. di  $A$ ),  
in particolare se  $\Lambda(A) \subseteq \text{LHP}$  o  $\Lambda(A) \subseteq \text{RHP}$

## Lyapunov equation: positivity

$$\operatorname{Re} \lambda_i < 0$$

### Lemma

Suppose  $A$  has eigenvalues in the (open) LHP. Then,  ~~$Q \succ 0$~~   $Q \succeq 0$  implies  $X \succeq 0$ , and  $Q \succ 0$  implies  $X \succ 0$ .

**Proof** Check that

$$X = \int_0^{\infty} e^{A^*t} Q e^{At} dt$$

(via  $\frac{d}{dt} e^{A^*t} Q e^{At} = \dots$ ).

### Lemma

Suppose  $Q \succ 0$  and  $X \succ 0$ . Then,  $A$  has eigenvalues in the (open) LHP.

**Proof** Let  $Av = \lambda v$ ; then  $0 < v^* Q v = \dots$

Alternative statement:  $A$  has all its eigenvalues in the LHP if there exists  $X \succ 0$  such that  $A^* X + X A \prec 0$ .

$$A^*X + XA + Q = 0$$

Lemma: se  $\lambda(A) \subseteq \text{LHP}$ ,

$$X = \int_0^{\infty} e^{tA^*} Q e^{tA} dt$$

(noto che è necessario  
che  $\lambda(A) \subseteq \text{LHP}$   
perché l'integrale converga  
per ogni  $Q$ )

(se  $\lambda(A) \subseteq \text{LHP}$ ,  $\exp(tA) \rightarrow 0$  esponenzialmente)

(si vede passando a blocchi di Jordan:  $A = VJV^{-1}$ ,

$$\exp(tA) = V \exp(tJ) V^{-1}$$

## Lyapunov's version

Alternative way to see it: the dynamical system  $\dot{x}(t) = Ax(t)$  is stable (i.e.,  $\lim_{t \rightarrow \infty} x(t) = 0$ ) for all initial values if and only if there exists  $X \succ 0$  such that  $A^*X + XA \prec 0$ .

Lyapunov's proof:  $E(t) = x(t)^* X x(t)$  ('energy function') is such that  $\dot{E}(t) \leq 0$ .

## Discrete-time version

Many of these results also come in a 'discrete-time' variant; in this case:

### Stein's equation

$A$  has all its eigenvalues in the (open) unit disc iff

$$X - A^*XA = Q, \quad Q \succ 0$$

has a solution  $X \succ 0$ .

Deals with stability of the discrete-time system  $x_{t+1} = Ax_t$ .

## Discrete-time version

Discrete-time version of the integral formula:

If  $A$  has all its eigenvalues inside the (open) unit disc then

$$X = \sum_{k=0}^{\infty} (A^*)^k Q A^k.$$

**Proof**  $(I - A^T \otimes A^*) \text{vec}(X) = \text{vec}(Q)$ , series for the inverse.