Example: control theory

$XAx + Bx + XC + D = 0$

Control theory (important subject in engineering) is the study of dynamical systems + controllers.

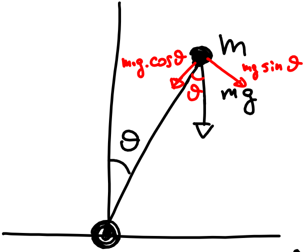
Example can we keep an 'inverted pendulum' in the upright position by applying a steering force?

State $x(t) = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, where θ is the angle formed by the pendulum (12 o' clock $\leftrightarrow \theta = 0$).

Free system equations:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ mg \sin x_1 \end{bmatrix} \approx \begin{bmatrix} x_2 \\ mgx_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ mg & 0 \end{bmatrix} x.$$

The system is not stable: $A = \begin{bmatrix} 0 & 1 \\ mg & 0 \end{bmatrix}$ has one positive and one negative eigenvalue.



motore che permette di applicare una forza

$$\cancel{m}g\cdot\sin\vartheta = F = m\cdot a = \cancel{m}\cdot\ddot{\vartheta}$$

Stato: $x = \begin{bmatrix}\vartheta \\ \dot{\vartheta}\end{bmatrix}$

$$x(t) = \begin{bmatrix}\vartheta(t) \\ \dot{\vartheta}(t)\end{bmatrix}$$

Equazione differenziale:

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\dot{\vartheta} \\ \ddot{\vartheta}\end{bmatrix} = \begin{bmatrix}x_2 \\ g\cdot\sin x_1\end{bmatrix}$$

Aggiungendo l'azione del motorino:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ g\sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}$$

se il motorino applica una forza $u(t)$ al tempo t

Se x è piccolo (pendolo quasi verticale)

$\sin x_1 \approx x_1$.

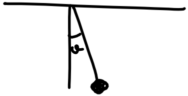
$$\boxed{\dot{x} = A \cdot x + B \cdot u} \qquad A = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Il sistema senza controllo $(u(t)=0)$ è stabile?

$\dot{x}=Ax$, dipende dagli autovalori di A

$A=\begin{bmatrix}0 & 1\\ g & 0\end{bmatrix}$ $\qquad \lambda_{12}=\pm\sqrt{g}$

(se avessi fatto il conto per il pendolo in basso,



mi sarebbe venuto $A=\begin{bmatrix}0 & 1\\ -g & 0\end{bmatrix}$, $\lambda_{12}=\pm i\sqrt{g}$

$\exp(A)$ ha autoval. di modulo 1 $\Rightarrow$ $x(t)$ limitata)

Example: controlling an inverted pendulum

$u : [0, \infty) \to \mathbb{R}$

$x : [0, \infty) \to \mathbb{R}^2$

Now we apply an additional steering force u:

$$\dot{x} = Ax + Bu, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Can we choose $u(t)$ so that the system is stable? Yes — even better: we can choose $u(t) = \underset{\sim}{F}x(t)$.

I.e., we can literally build a contraption (engine + camera) that sets the appropriate force according to the current state only (feedback control). $u = \begin{bmatrix} f_1 & f_2 \end{bmatrix} x$ gives

$$\dot{x} = (A + BF)x = \begin{bmatrix} 0 & 1 \\ f_1 + mg & f_2 \end{bmatrix} x.$$

Choosing f_1, f_2 appropriately we can move the eigenvalues of $A + BF$ arbitrarily.

$F = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$ $\quad u = Fx$

$$\dot{x} = Ax + Bu = (A + BF)x$$

stabilità dipende dagli autovalori di $A+BF =$

$$= \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g+f_1 & f_2 \end{bmatrix}$$

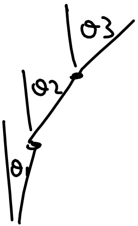
polin. caratteristico: $x(x-f_2) - (g+f)_1 =$

$$= x^2 - x f_2 - (g + f_1).$$

$\Rightarrow$ posso ottenere (scegliendo F opportunamente) due autovalori a mia scelta

Step 1: riscrivo il mio sistema dinamico (linearizzando rispetto a un punto di equilibrio $x=0$) come

$$\dot{x} = Ax + Bu$$



$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \qquad \begin{bmatrix} & \\ B & \\ & \end{bmatrix}$$

The general setup

dim. dello stato

numero di "controlli" indipendenti

$$\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n \times n}, \; B \in \mathbb{R}^{n \times m}.$$

Can we always stabilize a system? No — counterexample:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

No matter what we choose, we cannot change the dynamics of the second block of variables. If A_{22} has eigenvalues outside the LHP, there is nothing we can do.

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} A_{11}+B_1F_1 & A_{12}+B_1F_2 \\ 0 & \boxed{A_{22}} \end{bmatrix}$$

$\Rightarrow$ non esiste un "feedback control" $u = Fx$

(e neppure un controllo $u(t)$ qualunque, perché $\longrightarrow$

Scrivendo $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, le eq. diff.

sono

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + B_1 u \\ A_{22}x_2 \end{bmatrix}$$

→ non posso influenzare x_2, evolve sempre

come $\dot{x}_2 = A_{22}x_2$

Controllability / Stabilizability

This structure may be 'hidden' behind a change of basis, for instance $A \leftarrow KAK^{-1}, B \leftarrow KB$.

How do we check for it? Krylov spaces:

The pair (A, B) is called controllable if

$$\operatorname{span}(B, AB, \dots, A^k B, \dots) = \mathbb{R}^n.$$

The pair (A, B) is called stabilizable if

$$(KAK^{-1}, KB) = \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right)$$

with (A_{11}, B_1) controllable and A_{22} stable.

$$\operatorname{span}(B, AB, \dots, A^k B, \dots) = \mathbb{R}^n$$

<u>a meno che</u> io riesca a scrivere

$$A = K \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} K^{-1} \quad , \quad B = K \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

(col secondo blocco non banale).

(corrisponde a $\hat{x} \to Kx$

$$K\dot{\hat{x}} = K\dot{x} = K(Ax + Bu) = KAK^{-1}\hat{x} + KBu$$)

Freccia 1: se $A = K \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} K^{-1}$, $B = K \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$,

allora $A^i B = K \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \cancel{K^{-1}K} \begin{bmatrix} * \\ 0 \end{bmatrix} = K \begin{bmatrix} * \\ 0 \end{bmatrix} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} * \\ 0 \end{bmatrix} = \operatorname{span} K_1$

Supponiamo che $\mathrm{span}(B, AB, \dots A^iB, \dots) \neq \mathbb{R}^n$

Allora, $\exists\ v \neq 0$ tale che

$$v^*B = v^*AB = v^*A^2B = \dots = 0$$

Cambio base in modo che $v = [0\ 0 \cdots 0\ 1]$.

Allora, $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$ $\quad AB = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 \\ A_{21}B_1 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}$

$\Rightarrow A_{21}B_1 = 0$

M: basta dimostrare che $v^*b = v^*Ab = v^*A^2b = \dots = 0$ per $b \in \mathbb{R}^n$.

Bass algorithm

$A+\alpha I$ ha autoval. $\lambda_i+\alpha$ (se λ_i autoval. di A)
$\alpha \in \mathbb{R}$ abbastanza grande $\Rightarrow$ $Re(\lambda_i+\alpha)>0$

Let $\alpha > \rho(A)$; then $A + \alpha I$ has eigenvalues in the RHP, and the Lyapunov equation

$$(A+\alpha I)X + X(A+\alpha I)^* = 2BB^*$$

has a solution $X \succeq 0$.

We shall show that $X \succ 0$ (whenever (A, B) controllable). Then,

$$(A - BB^*X^{-1})X + X(A - BB^*X^{-1})^* = -2\alpha X,$$

which proves that $\underbrace{A - B(B^*X^{-1})}$ has eigenvalues in the LHP.

$F = -B^* X^{-1}$ stabilizza il sistema

(Actually, if (A, B) is controllable, we can find F such that $A + BF$ has any chosen spectrum.)

$$(A+\alpha I)X + X(A+\alpha I)^* = \underline{2BB^*} \qquad (1)$$

$$(-A-\alpha I)X + X(-A-\alpha I)^* + 2BB^* = 0$$

$-A-\alpha I$ ha autovalori nel LHP, $2BB^* = Q \geq 0$

$\Rightarrow X \geq 0$

In realtà, possiamo dimostrare che se (A, B) controllabili allora $X > 0$

Da (1) segue che

$$\boxed{\underbrace{(A-BB^*X^{-1})X}_{= AX - BB^*} + \underbrace{X(A-BB^*X^{-1})^*}_{= X(A^*-X^{-1}BB^*) = XA^*-BB^*} + 2\alpha X = 0} \qquad (2)$$

(2) è un'equazione di Lyapunov,

Il termine noto $Q = 2\alpha X$ è > 0

La soluzione è $X > 0$

$\Rightarrow$ $A - BB^* X^{-1}$ ha tutti gli autovalori nel LHP.

È un caso particolare di (3) nella prossima slide, con $\hat{B} = \sqrt{2}B$, $\hat{A} = A + \alpha I$

(nota che $\mathrm{span}\left(B, AB, A^2 B, \dots\right)$

$$= \mathrm{span}\left(B, (A+\alpha I)B, (A+\alpha I)^2 B, \dots\right),$$

quindi (A,B) contr. $\Leftrightarrow$ $(A+\alpha I, B)$ contr.

Controllability Lyapunov equation

Let A be a stable matrix. (A, B) is controllable iff the solution of

$$AX + XA^* = BB^* \qquad (3)$$

is positive definite.

Proof $\Rightarrow$ suppose (A, B) is not controllable. Then, (up to a change of basis) (e X_1 risolve $A_{11}X_1 + X_{11}A_{11}^* = B_1B_1^*$)

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}^* & 0 \\ A_{12}^* & A_{22}^* \end{bmatrix} = \begin{bmatrix} B_1 B_1^* & 0 \\ 0 & 0 \end{bmatrix}.$$

so X is not posdef.

$\Leftarrow$ Suppose (A, B) is controllable. Then, for each $v \neq 0$, $v^* A^k B$ is not zero for all $k \implies v^* e^{At} B$ is not zero for all $t \implies v^* X v = \int v^* e^{At} B B^* e^{A^* t} v \mathrm{d}t \neq 0$.

Se $v^* A^k B = 0 \quad \forall k \in \mathbb{N}$, allora

$$v^* \cdot \mathrm{span}(B, AB, A^2B \dots) = 0$$

e quindi lo span non è tutto $\mathbb{R}^n$

$\overset{=0}{v^*} e^{At} B$ non è zero per ogni t

$$\parallel$$

$$v^* \underbrace{\left(I + At + \frac{1}{2} A^2 t^2 + \dots\right) B}$$

Ip: $v^* \exp(tA) B = 0 \qquad \forall t \in [0, \infty]$

Tesi: $v^* A^k B = 0 \qquad \forall k \in \mathbb{N}$

$\boxed{k=0}$ $0 = v^* \exp(0 \cdot A) B = v^* B$

$\boxed{k=1}$ $\exp(tA) = I + tA + o(t^2) \rightarrow \lim_{t \to 0} \frac{\exp(tA) - I}{t} = A$

$$0 = \lim_{t \to 0} \underbrace{v^* \left(\frac{\exp tA - I}{t} \right) B}_{= 0} = v^* AB$$

$\boxed{k=2}$ $\exp(tA) = I + tA + \frac{t^2}{2} A^2 + O(t^3)$

$$0 = \lim_{t \to 0} \underbrace{v^* \frac{\exp(tA) - I - tA}{t^2} B}_{\substack{\parallel \\ 0}} = v^* A^2 B$$

$$\vdots$$

Caso più facile che possiamo fare con altri metodi:

se $m=1$, $B=b$ è un singolo vettore.

Se (A,b) è controllabile, prendo la base

$$M=\left[b, Ab, A^2 b, \dots A^{n-1}b\right]$$

di $\mathbb{R}^n$; in questa base

$$M^{-1}AM=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha_0 \\ 1 & 0 & 0 & & & \alpha_1 \\ 0 & 1 & 0 & \ddots & & \vdots \\ 0 & 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 & \alpha_{n-1} \end{bmatrix} \qquad M^{-1}b=\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A^n b = A\cdot A^{n-1}b = \sum_{i=0}^{n-1} \alpha_i A^i b$$