Example: control theory

Control theory (important subject in engineering) is the study of dynamical systems + controllers.

Example can we keep an 'inverted pendulum' in the upright position by applying a steering force?

State $x(t) = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, where θ is the angle formed by the pendulum (12 o' clock $\leftrightarrow \theta = 0$).

Free system equations:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ mg \sin x_1 \end{bmatrix} \approx \begin{bmatrix} x_2 \\ mgx_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ mg & 0 \end{bmatrix} x.$$

The system is not stable: $A = \begin{bmatrix} 0 & 1 \\ mg & 0 \end{bmatrix}$ has one positive and one negative eigenvalue.

Example: controlling an inverted pendulum

Now we apply an additional steering force u:

$$\dot{x} = Ax + Bu, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Can we choose $u(t)$ so that the system is stable? Yes — even better: we can choose $u(t) = Fx(t)$.

I.e., we can literally build a contraption (engine + camera) that sets the appropriate force according to the current state only (feedback control). $u = \begin{bmatrix} f_1 & f_2 \end{bmatrix} x$ gives

$$\dot{x} = (A + BF)x = \begin{bmatrix} 0 & 1 \\ f_1 + mg & f_2 \end{bmatrix} x.$$

Choosing f_1, f_2 appropriately we can move the eigenvalues of $A + BF$ arbitrarily.

The general setup

$$\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}.$$

Can we always stabilize a system? No — counterexample:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

No matter what we choose, we cannot change the dynamics of the second block of variables. If A_{22} has eigenvalues outside the LHP, there is nothing we can do.

Controllability / Stabilizability

This structure may be 'hidden' behind a change of basis, for instance $A \leftarrow KAK^{-1}, B \leftarrow KB$.

How do we check for it? Krylov spaces:

The pair (A, B) is called controllable if

$$\operatorname{span}(B, AB, \ldots, A^k B, \ldots) = \mathbb{R}^n.$$

The pair (A, B) is called stabilizable if

$$(KAK^{-1}, KB) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

with (A_{11}, B) controllable and A_{22} stable.

Bass algorithm

Let $\alpha > \rho(A)$; then $A + \alpha I$ has eigenvalues in the RHP, and the Lyapunov equation

$$(A + \alpha I)X + X(A + \alpha I)^* = 2BB^*$$

has a solution $X \succeq 0$.

We shall show that $X \succ 0$ (whenever (A, B) controllable). Then,

$$(A - BB^*X^{-1})X + X(A - BB^*X^{-1})^* = -2\alpha X,$$

which proves that $A - B(B^*X^{-1})$ has eigenvalues in the LHP.

(Actually, if (A, B) is controllable, we can find F such that $A + BF$ has any chosen spectrum.)

Controllability Lyapunov equation

Let A be a stable matrix. (A, B) is controllable iff the solution of

$$AX + XA^* = BB^*$$

is positive definite.

Proof $\Rightarrow$ suppose (A, B) is not controllable. Then, (up to a change of basis)

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}^* & 0 \\ A_{12}^* & A_{22}^* \end{bmatrix} = \begin{bmatrix} B_1 B_1^* & 0 \\ 0 & 0 \end{bmatrix}.$$

so X is not posdef.
$\Leftarrow$ Suppose (A, B) is controllable. Then, for each $v \neq 0$, $v^* A^k B$ is not zero for all $k \implies v^* e^{At} B$ is not zero for all $t \implies$ $v^* X v = \int v^* e^{At} BB^* e^{A^* t} v \mathrm{d}t \neq 0$.