Controllability

Definition

$(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is controllabile iff $K(A, B) = \mathbb{R}^n$, where

$$K(A, B) := \operatorname{span}(B, AB, A^2B, \dots).$$

Lemma

There exists a nonsingular $M \in \mathbb{R}^{n\times n}$ such that

$$M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

(with $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times m}$, and $n_2 \neq 0$) if and only if (A, B) is not controllable.

Proof

$\Rightarrow$ Partition $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$ conformably. Then,
$A^k B = M \begin{bmatrix} A_{11}^k B_1 \\ 0 \end{bmatrix} = M_1 A_{11}^k B_1$, so $K(A, B) \subseteq \operatorname{Im} M_1$.

$\Leftarrow$ Let the columns of M_1 be a basis of $K(A, B)$, and complete it to a nonsingular $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$. Then, $M^{-1}AM$ is block triangular (because M_1 is A-invariant), and $M^{-1}B$ has zeros in the second block row (because the columns of B lie in $\operatorname{Im} M_1$).

(Linear algebra characterization: $K(A, B)$ is the smallest A-invariant subspace that contains B. It's the space Q_n that we obtain after we encounter breakdown in Arnoldi.)

Kalman decomposition

Kalman decomposition

For every matrix pair $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$, there is a change of basis M such that

$$M^{-1}AM = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad M^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

with (A_{11}, B_1) controllable.

Proof: as above: take M_1 such that its columns are a basis of the 'controllable space' $K(A, B)$, then complete it to a basis of $\mathbb{R}^n$.

Stabilizability

Definition

(A, B) is stabilizable if in its Kalman decomposition A_{22} is stable (i.e., $\Lambda(A_{22}) \subseteq LHP$).

Note that this definition is well-posed even if M is non-unique: the eigenvalues of A_{11} are the eigenvalues of $A|_{K(A,B)}$, and those of A_{22} are the remaining eigenvalues of A (counting with their algebraic multiplicity).

Controllability Lyapunov equation

Theorem

If A is a stable matrix and (A, B) is controllable, then the solution of $AX + XA^* + BB^* = 0$ is positive definite.

Proof We know already that $X = \int_0^\infty \exp(At)BB^* \exp(A^*t)\,\mathrm{d}t \succeq 0$.

Let $v \neq 0$ be any vector. We need to show that $v^*Xv \neq 0$.

If $v^* \exp(tA)B \neq 0$ for some t, then we are done (it is nonzero in a neighbourhood by continuity...).

If $v^* \exp(tA)B = 0$ for all t, then $v^*B = 0$ (taking $t = 0$),

$$\lim_{t\to 0} \frac{1}{t} v^*(\exp(tA) - I)B = v^*Ab = 0,$$

$$\lim_{t\to 0} \frac{1}{t^2} v^*(\exp(tA) - I - tA)B = v^* \frac{1}{2} A^2 b = 0,$$

$$\vdots$$