Optimal control

Several choices for stabilizing — for instance, you can choose different αs in Bass's algorithm.

Is there an 'optimal' one?

Linear-quadratic optimal control

Find $u : [0, \infty] \to \mathbb{R}$ (piecewise C^0, let's say) that minimizes

$$E = \int_0^\infty x^T Q x + u^T R u \, \mathrm{d}t$$

$$\text{s.t. } \dot{x} = Ax + Bu, \; x(0) = x_0.$$

Minimum 'energy' defined by a quadratic form ($R \succeq 0$, $Q \succeq 0$).

We assume $R \succ 0$: control is never free. Trickier problem otherwise.

Optimal control — solution

Using calculus of variations tools, one can prove that (Pontryagin's maximum principle)

A pair of functions u, x solves the optimal control problem iff there exists a function $\mu(t)$ ('Lagrange multiplier') such that

$$\begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^T & Q & 0 \\ B^T & 0 & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix},$$

$x(0) = x_0$, $\lim_{t\to\infty} \left[\begin{smallmatrix} \mu \\ x \\ u \end{smallmatrix}\right] = 0$.

Structure of the problem

$$\mathcal{E}\begin{bmatrix}\dot{\mu}\\ \dot{x}\\ \dot{u}\end{bmatrix} = \mathcal{A}\begin{bmatrix}\mu\\ x\\ u\end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix}0 & A & B\\ A^T & Q & 0\\ B^T & 0 & R\end{bmatrix}, \mathcal{E} = \begin{bmatrix}0 & I & 0\\ -I & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Pencils $\lambda\mathcal{E} - \mathcal{A}$ with $\mathcal{A} = \mathcal{A}^T$, $\mathcal{E} = -\mathcal{E}^T$ are called even.

Eigenvalue pairing: if $(\lambda\mathcal{E} - \mathcal{A})v = 0$, then $v^T(-\overline{\lambda}\mathcal{E} - \mathcal{A}) = 0$, and $-\overline{\lambda}$ is an eigenvalue, too.

On a real problem, eigenvalues usually come in quadruples, $(\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda})$. They may be degenerate if λ is real or pure imaginary.

The eigenvalues

If $R \succ 0$, row/column operations give

$$\lambda \mathcal{E} - \mathcal{A} \sim \lambda \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -BR^{-1}B^T & A & 0 \\ A^T & Q & 0 \\ 0 & 0 & I \end{bmatrix}.$$

This shows that $\lambda \mathcal{E} - \mathcal{A}$ has m simple eigenvalues at ∞, plus $2n$ finite eigenvalues (with multiplicity): those of

$$\begin{bmatrix} & I \\ -I & \end{bmatrix}^{-1} \begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix}.$$

Change of variables

The same idea, recast as a change of variables on the equations:
μ, x, u solve

$$\begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^T & Q & 0 \\ B^T & 0 & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix}$$

iff $u = -R^{-1}B^T\mu$ and μ, x solve

$$\begin{bmatrix} & I \\ -I & \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -BR^{-1}B^T & A \\ A^T & Q \end{bmatrix} \begin{bmatrix} \mu \\ x \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ \mu \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix}, \quad G = BR^{-1}B^T.$$

Solving the reduced problem

Suppose that:

- $\mathcal{H}$ has n eigenvalues in the LHP and n in the RHP. (Recall: $\mathcal{H}$ has "even eigensymmetry").
- we find X such that $\begin{bmatrix} I \\ X \end{bmatrix}$ spans the stable (eigenvalues $\in LHP$) invariant subspace of $\mathcal{H}$, i.e., $\mathcal{H} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{R}$.

Then, the stable solutions of

$$\begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ \mu \end{bmatrix}$$

are given by

$$\begin{bmatrix} x(t) \\ \mu(t) \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \exp(\mathcal{R}t)v.$$

The initial condition $x(0) = x_0$ gives $v = x_0$. Moreover, $\mu(t) = Xx(t)$, hence $u(t) = -R^{-1}B^T Xx(t)$.

Algebraic Riccati equations

We have reduced the problem to $\mathcal{H} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{R}$, or

$$\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{R}$$

$$\mathcal{R} = A - GX, \quad -Q - A^T X = XA - XGX.$$

$$A^T X + XA + Q - XGX = 0, \quad Q \succeq 0, \; G \succeq 0$$

is called algebraic Riccati equation.

We look for a stabilizing solution, i.e., $\Lambda(\mathcal{R}) \subseteq LHP$.

(Note that $\Lambda(\mathcal{R}) \subset \Lambda(\mathcal{H})$.)

Next goal: show that we can do what we claimed in the previous slide.

Solvability conditions

Solutions of (ARE) $\iff$ n-dimensional invariant subspaces of $\mathcal{H}$ with invertible top block.

If $\mathcal{H}$ has distinct eigenvalues, there are at most $\binom{2n}{n}$ solutions (choose n eigenvalues out of the $2n\dots$)

Does it have a (unique) stabilizing solution? $\mathcal{H}$ Must have (exactly) n eigenvalues in the LHP, and the associated invariant subspace must be expressible as $\operatorname{Im}\begin{bmatrix} I \\ X \end{bmatrix}$.

Hamiltonian matrices

$$\mathcal{H} = \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}, \quad Q = Q^*,\ G = G^*$$

is a Hamiltonian matrix, i.e., it satisfies $J\mathcal{H} = -\mathcal{H}^* J$, where $J = \begin{bmatrix} & I \\ -I & \end{bmatrix}$.

(Skew-self-adjoint with respect to the antisymmetric scalar product defined by J.)

If $\mathcal{H}v = \lambda v$, then $(v^* J)\mathcal{H} = (-\overline{\lambda})(v^* J)$: eigenvalues have 'even symmetry', and the right eigenvector relative to λ is related to the left one relative to $-\overline{\lambda}$.

A similar relation can be proved for Jordan chains: λ and $-\overline{\lambda}$ have Jordan chains of the same size.

Solvability conditions

Theorem

Assume $Q \succ 0$, and (A, B) stabilizable. Then, $\mathcal{H}$ has no eigenvalues with $\operatorname{Re} \lambda = 0$.

($Q \succ 0$ can be weakened to $Q \succeq 0$ and (A^T, Q^T) stabilizable.)

Proof (sketch)

Suppose instead $\mathcal{H}\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = \imath\omega\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$; from $0 = \operatorname{Re}\left[\begin{smallmatrix} z_2^* & z_1^* \end{smallmatrix}\right]\left[\begin{smallmatrix} A & -G \\ -Q & -A^* \end{smallmatrix}\right]\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = z_2^* G z_2 + z_1^* Q z_1$ follows that $z_1 = 0$, $z_2^* B = 0$. But the latter together with $z_2^* A = -\imath\omega z_2^*$ contradicts stabilizability.

Hence, $\mathcal{H}$ has n eigenvalues in the LHP and n associated ones in the RHP: it has exactly one stabilizing subspace.

Form of the invariant subspace

We know now that there exist $U_1, U_2 \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ spans the stable invariant subspace.
Moreover, $\begin{bmatrix} U_1^* & U_2^* \end{bmatrix} J = \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix}$ spans the left anti-stable invariant subspace.

Left and right invariant subspaces relative to disjoint eigenvectors are orthogonal $\implies$

$$0 = \begin{bmatrix} U_2^* & -U_1^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = U_2^* U_1 - U_1^* U_2.$$

We'd like to show that U_1 is invertible. Then (up to changing basis in $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$) we can take $U_1 = I$, $U_2 = X = X^*$.

Nonsingularity of U_1

Suppose (A, B) stabilizable, $Q \succeq 0$, $G \succeq 0$. Then U_1 is invertible.

We'd like to show that U_1 is nonsingular. Suppose otherwise $U_1 v = 0$, $U_2 v \neq 0$. Then,

$$-v^* U_2^* G U_2 v = \begin{bmatrix} v^* U_2^* & 0 \end{bmatrix} \mathcal{H} \begin{bmatrix} 0 \\ U_2 v \end{bmatrix} = v^* \begin{bmatrix} U_2^* & U_1^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \mathcal{R} = 0.$$

implies $BU_2 v = 0$.
The first block row gives $U_1 \mathcal{R} v = 0 \implies U_1$ is $\mathcal{R}$-invariant and there are x, $\lambda \in LHP$ such that $U_1 x = 0$, $\mathcal{R} x = \lambda x$. Now the second block rows gives $-A^T U_2 x = \lambda U_2 x$. This (together with $BU_2 x = 0$ from above) contradicts stabilizability.

How to solve Riccati equations

- Newton's method.
- Invariant subspace via unstructured methods (QR).
- Invariant subspace via 'semi-structured' methods (Laub trick).
- Invariant subspace via structured methods (URV).
- Doubling / Sign iteration.