Newton's method for CARE

$$F(X) = A^* X + XA + Q - XGX$$

$$L_{F,X}(E) = A^* E + EA - EGX - XGE = E(A - GX) + (A - GX)^* E.$$

$$\widehat{L}_{F,X} = (A - GX)^T \otimes I + I \otimes (A - GX)^*.$$

If X_* is the stabilizing solution then $\Lambda(A - GX_*) \subset LHP \implies L_{F,X_*}$ is nonsingular.

Newton's method

For $k = 0, 1, 2, \ldots$

1. Solve $E(A - GX_k) + (A - GX_k)^* E = F(X_k)$ for E;
2. Set $X_{k+1} = X_k - E$.

Newton's method

Note that $E(A - GX_k) + (A - GX_k)^* E = F(X_k)$ is equivalent to

$$X_{k+1}(A - GX_k) + (A - GX_k)^* X_{k+1} = -Q - X_k G X_k.$$

This shows that $A - GX_k$ stable $\implies X_{k+1} \succeq 0$.

Actually, something stronger holds.

Monotonicity of Newton's method

Theorem

Suppose X_0 is chosen such that $\Lambda(A - GX_0) \subset LHP$. Then, $X_1 \succeq X_2 \succeq X_3 \succeq \cdots \succeq X_* \succeq 0$. Moreover, $X_k \to X_*$ quadratically.

Proof (sketch) Coupled induction. Set $A_k := A - GX_k$:

$$(X_k - X_{k+1})A_k + A_k^*(X_k - X_{k+1}) = -(X_k - X_{k-1})G(X_k - X_{k-1})$$
$$(X_* - X_{k+1})A_k + A_k^*(X_* - X_{k+1}) = -(X_* - X_k)G(X_* - X_k)$$

hence A_k stable $\implies X_k \succeq X_{k+1} \succeq X_*$.

$$(X_{k+1} - X_*)A_{k+1} + A_{k+1}^*(X_{k+1} - X_*)$$
$$= -(X_{k+1} - X_k)G(X_{k+1} - X_k) - (X_{k+1} - X_*)G(X_{k+1} - X_*)$$

This does not prove immediately that A_{k+1} is stable (because the RHS is not $\prec 0$), but $A_{k+1}v = \lambda v$ with $\operatorname{Re}\lambda \geq 0$ gives $DX_{k+1}v = DX_k v$, hence if $A_k v = \lambda v$.

Newton: wrap-up

- Use Bass's algorithm to find X_0 such that $A - GX_0$ is stable
- Run Newton iterations till convergence.

Expensive: each iteration requires a Schur form.

One final step of Newton can be used to 'correct' an inaccurate algorithm.