Newton's method for CARE

$$F(X) = A^*X + XA + Q - XGX$$

$$L_{F,X}(E) = A^*E + EA - EGX - XGE = E(A - GX) + (A - GX)^*E.$$

$$\widehat{L}_{F,X} = (A - GX)^T \otimes I + I \otimes (A - GX)^*.$$

If X_* is the stabilizing solution then $\Lambda(A - GX_*) \subset LHP \implies L_{F,X_*}$ is nonsingular.

Newton's method

For $k = 0, 1, 2, \ldots$

1. Solve $E(A - GX_k) + (A - GX_k)^*E = F(X_k)$ for E;
2. Set $X_{k+1} = X_k - E$.

$$F(X) = A^*X + XA + Q - XGX$$

$$A^*(X+E) + (X+E)A + Q - (X+E)G(X+E) - A^*X - XA - Q + XGX$$

$$= \underbrace{A^*E + EA - EGX - XGE}_{L_{F,X}(E)} + o(\|E\|)$$

$$= E(A-GX) + (A-GX)^*E$$

$$\hat{L}_{F,X} = (A-GX)^T \otimes I + I \otimes (A-GX)^*$$

Se X_* è la soluzione stabilizzante della ARE $F(X)=0$, $A-GX_*$ è stabile e ha autoval. uguali a quelli stabili di $\mathcal{H}$.

$\Rightarrow$ Gli autoval. di

L_{F,X_*} sono $\lambda_i+\lambda_j$, dove λ_i sono gli autoval. di $A-GX_*$

$\lambda_i+\lambda_j$ ha sempre parte reale < 0.

Metodo di Newton:

$X_{k+1}=X_k-E$, dove E risolve $L_{F,X_k}(E)=F(X_k)$, cioè

$$(A-GX_k)^*E+E(A-GX_k)=\underbrace{Q+A^*X_k+X_kA-X_kGX_k} \qquad \ast$$

(equazione di Lyapunov)

$$(A-GX_k)^*X_k + X_k(A-GX_k) = \underline{A^*X_k + XA - 2X_kGX_k} \quad (**)$$

Sottraggo $(*)$ da $(**)$, e viene

$$(A-GX_k)^*\underbrace{(X_k-E)}_{=X_{k+1}} + \underbrace{(X_k-E)}_{=X_{k+1}}(A-GX_k) = -X_kGX_k - Q \leqslant 0$$

$$\Downarrow$$

$$-(X_k^*)GX_k \leqslant 0,$$

ottenuta coniugando G

$\Rightarrow$ Se $A-GX_k$ ha autoval. nel LHP, allora $X_{k+1} \geqslant 0$.

Remark: se X_* è la stabilizing solution della ARE,

$$(A-GX_*)^* X_* + X_*(A-GX_*) = -Q - X_*GX_*$$

$$\Updownarrow$$

$$A^*X_* - X_*GX_* + X_*A - \cancel{X_*GX_*} = -Q - \cancel{X_*GX_*}$$

Quindi X_* risolve l'eq. di Lyapunov

$$(A-GX_*)^* Z + Z(A-GX_*) = -Q - X_*GX_*$$

e $A-GX_*$ è stabile $\Rightarrow X_* \succeq 0$.

Newton's method

Note that $\boxed{E(A - GX_k) + (A - GX_k)^* E = F(X_k)}$ is equivalent to

$$X_{k+1}(A - GX_k) + (A - GX_k)^* X_{k+1} = -Q - X_k G X_k.$$

This shows that $A - GX_k$ stable $\implies X_{k+1} \succeq 0$.

Actually, something stronger holds.

Monotonicity of Newton's method

facile, perché è Newton

si parte da 1: $X_0 \succeq X_1$ non sempre vero

Theorem

Suppose X_0 is chosen such that $\Lambda(A - GX_0) \subset LHP$. Then, $X_1 \succeq X_2 \succeq X_3 \succeq \cdots \succeq X_* \succeq 0$. Moreover, $X_k \to X_*$ quadratically.

Proof (sketch) Coupled induction. Set $\boxed{A_k := A - GX_k:}$

$$(X_k - X_{k+1})A_k + A_k^*(X_k - X_{k+1}) = -(X_k - X_{k-1})G(X_k - X_{k-1})$$
$$(X_* - X_{k+1})A_k + A_k^*(X_* - X_{k+1}) = -(X_* - X_k)G(X_* - X_k)$$

hence A_k stable $\implies X_k \succeq X_{k+1} \succeq X_*$.

$$\begin{aligned}(X_{k+1} - X_*)A_{k+1} &+ A_{k+1}^*(X_{k+1} - X_*)\\ &= -(X_{k+1} - X_k)G(X_{k+1} - X_k) - (X_{k+1} - X_*)G(X_{k+1} - X_*)\end{aligned}$$

This does not prove immediately that A_{k+1} is stable (because the RHS is not $\prec 0$), but $A_{k+1}v = \lambda v$ with $\operatorname{Re}\lambda \geq 0$ gives $DX_{k+1}v = DX_k v$, hence if $A_k v = \lambda v$.

$$A_k := A - GX_k$$

Newton: wrap-up

- Use Bass's algorithm to find X_0 such that $A - GX_0$ is stable
- Run Newton iterations till convergence.

Expensive: each iteration requires a Schur form.

One final step of Newton can be used to 'correct' an inaccurate algorithm.