# Invariant subspace methods for CAREs

$X$ solves CARE $A^*X + XA + Q = XGX$ iff

$$\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{R}, \quad \mathcal{R} = A - GX.$$

One can find $X$ through an invariant subspace of the Hamiltonian.

```
>> [A,G,Q] = carex(4) %if test suite is installed
>> n = length(A);
>> H = [A -G; -Q -A'];
>> [U, T] = schur(H);
>> [U, T] =ordschur(U, T, 'lhp');
>> X = U(n+1:2*n, 1:n) / U(1:n, 1:n);
```

People are not satisfied with this method though — it is not structured backward stable.

Eigenvalues close to the imaginary axis can be ‘mixed up’ — try `carex(14)` for instance.

# Symplectic transformations

Ideal setting: make transformations at each step that are orthogonal and symplectic, i.e., orthogonal w.r.t the scalar product $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$: they satisfy $S^T JS = J$.

For instance:

- If $Q \in \mathbb{R}^{n\times n}$ is any orthogonal matrix, then blkdiag$(Q, Q)$ is orthogonal and symplectic.
- A Givens matrix that acts on entries $k$ and $n+k$ (i.e., `G = eye(2*n); G([k,n+k], [k,n+k]) = [c s; -s c];`) is orthogonal and symplectic.

Laub trick: let $U = \begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix}$ the unitary matrix produced by `schur(H)`. Then, $\left[\begin{smallmatrix} U_1 \\ U_2 \end{smallmatrix}\right]$ is an orthogonal matrix that spans the stable subspace. We know that $-J\left[\begin{smallmatrix} U_1 \\ U_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_2 \\ -U_1 \end{smallmatrix}\right]$ is orthogonal to it (and spans the left unstable invariant subspace).

It turns out that $V = \begin{bmatrix} U_1 & U_2 \\ U_2 & -U_1 \end{bmatrix}$ is orthogonal and symplectic, and $V^T \mathcal{H} V = \begin{bmatrix} R & S \\ 0 & -R^* \end{bmatrix}$, with $R$ upper triangular and $S$ symmetric (Hamiltonian Schur form).

# An orthogonal symplectic algorithm

This produces the same subspace as the previous method, so it is not really a 'structured' method. Can one do a 'symplectic QR' and compute the Hamiltonian Schur form using a sequence of orthosymplectic transformations?

Open problem for a while; it turns out that that Schur form does not exist for all Hamiltonian matrices (there are counterexamples with eigenvalues on the unit circle). $\implies$ algorithms must be unstable 'nearby'.

(This problem was known as Van Loan's curse.)

# Chu–Liu–Mehrmann algorithm

Closest thing to a solution: Chu–Liu–Mehrmann algorithm. Based on a different decomposition: $\mathcal{H} = URV^T$, with $U, V$ orthosymplectic and

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

with $R_{11}, R_{22}^*$ upper triangular.
Can be computed 'almost' directly in $O(n^3)$ (it's an LU-like decomposition).

# URV — simpler version (produces Hessenberg $R_{22}$)

- Left-multiply by blkdiag(Q, Q) to get $\begin{matrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{matrix}$
- Left-multiply by a Givens on $(1, n+1)$ to get $\begin{matrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{matrix}$
- Left-multiply by blkdiag(Q, Q) to get $\begin{matrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{matrix}$
- Right-multiply by blkdiag(Q, Q) to get $\begin{matrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{matrix}$
- Right-multiply by a Givens on $(2, n+2)$ to get $\begin{matrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{matrix}$
- Right-multiply by blkdiag(Q, Q) to get $\begin{matrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & 0 \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{matrix}$

# Using URV

Note that $\mathcal{H} = URV$ + symplecticity implies

$$\mathcal{H} = V \begin{bmatrix} -R_{22}^T & R_{12}^T \\ 0 & -R_{11}^T \end{bmatrix} U^T.$$

Hence

$$\mathcal{H}^2 = V \begin{bmatrix} -R_{11}R_{22}^T & * \\ 0 & -R_{22}R_{11}^T \end{bmatrix} V^T$$

can be used to compute eigenvalues (easily) and eigenvectors of $\mathcal{H}$ (for instance: the columns of $V$ cause breakdown at step 2 in Arnoldi).