Invariant subspace methods for CAREs

X solves CARE $A^*X + XA + Q = XGX$ iff

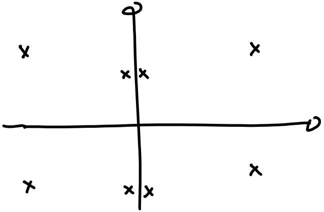
$$\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{R}, \quad \mathcal{R} = A - GX.$$

One can find X through an invariant subspace of the Hamiltonian.

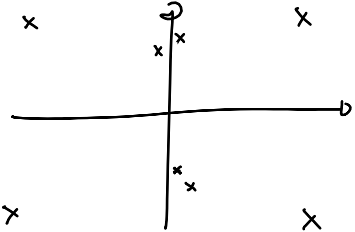
```
>> [A,G,Q] = carex(4) %if test suite is installed
>> n = length(A);
>> H = [A -G; -Q -A'];
>> [U, T] = schur(H);
>> [U, T] =ordschur(U, T, 'lhp');
>> X = U(n+1:2*n, 1:n) / U(1:n, 1:n);
```

People are not satisfied with this method though — it is not structured backward stable.

Eigenvalues close to the imaginary axis can be 'mixed up' — try `carex(14)` for instance.



Un metodo di Schur potrebbe produrre



Vorremmo un metodo che ci assicuri che gli autoval. hanno la struttura richiesta.

Se

$$\mathcal{H} = \begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix}^T,$$

allora $\mathrm{span}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ è un sottospazio invariante:
(associato agli autovalori di T_{11}).

$$\mathcal{H}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \underbrace{\begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix} \underbrace{\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \underbrace{\begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix}^T \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}}_{=\begin{bmatrix} I \\ 0 \end{bmatrix}}}_{=\begin{bmatrix} T_{11} \\ 0 \end{bmatrix}}}_{=\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} T_{11}}$$

$X = U_2 U_1^{-1}$

Symplectic transformations

Ideal setting: make transformations at each step that are orthogonal and symplectic, i.e., orthogonal w.r.t the scalar product $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$: they satisfy $S^T JS = J$.

For instance:

- If $Q \in \mathbb{R}^{n \times n}$ is any orthogonal matrix, then blkdiag(Q, Q) is orthogonal and symplectic.
- A Givens matrix that acts on entries k and $n + k$ (i.e., `G = eye(2*n); G([k,n+k], [k,n+k]) = [c s; -s c];`) is orthogonal and symplectic.

<u>Def</u>: $S \in \mathbb{C}^{2n \times 2n}$ si dice <u>simplettica</u> se

$$\langle v, w \rangle_J = \langle Sv, Sw \rangle_J \quad \text{per ogni } v, w \in \mathbb{C}^{2n}$$

cioè $v^* J w = v^* S^* J S w \quad \forall v, w \iff J = S^* J S$

$S^* = J S^{-1} J$

<u>Lemma</u>: se H Hamiltoniana, S <u>simplettica</u>, $S^{-1} H S$ è Hamiltoniana: $\boxed{JH = -H^* J}$

$$(S^{-1} H S)^* = S^* H^* S^{-*} = J S^{-1} J \, J H J \, J S J =$$

$$= J (S^{-1} H S) J$$

$$J(S^{-1} H S) = (S^{-1} H S)^* J^{-1} = -(S^{-1} H S) J$$

Laub trick: let $U = \begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix}$ the unitary matrix produced by (ord) `schur(H)`. Then, $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ is an orthogonal matrix that spans the stable subspace. We know that $-J\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_2 \\ -U_1 \end{bmatrix}$ is orthogonal to it (and spans the left unstable invariant subspace).

It turns out that $V = \begin{bmatrix} U_1 & U_2 \\ U_2 & -U_1 \end{bmatrix}$ is orthogonal and symplectic, and $V^T \mathcal{H} V = \begin{bmatrix} R & S \\ 0 & -R^* \end{bmatrix}$, with R upper triangular and S symmetric (Hamiltonian Schur form).

Prendo $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ prodotto dal metodo di Schur,

e ci attacco $J \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_2 \\ -U_1 \end{bmatrix}$,

ottenendo

$S = \begin{bmatrix} U_1 & U_2 \\ U_2 & -U_1 \end{bmatrix}$. Risultato: S è ortogonale e simplettico:

(perché $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ ha col. orto.)

1) $S^* S = \begin{bmatrix} U_1^* & U_2^* \\ U_2^* & -U_1^* \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ U_2 & -U_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

$U_1^* U_2 - U_2^* U_1$,

che è la relazione $\begin{bmatrix} U_1^* & U_2^* \end{bmatrix} J \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0$

2) $J \overset{?}{=} S^* J S$

$$S^* J S = \begin{bmatrix} U_1^* & U_2^* \\ U_2^* & -U_1^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ U_2 & -U_1 \end{bmatrix} =$$

$$= \begin{bmatrix} U_1^* & U_2^* \\ U_2^* & -U_1^* \end{bmatrix} \Big[$$

An orthogonal symplectic algorithm

This produces the same subspace as the previous method, so it is not really a 'structured' method. Can one do a 'symplectic QR' and compute the Hamiltonian Schur form using a sequence of orthosymplectic transformations?

Open problem for a while; it turns out that that Schur form does not exist for all Hamiltonian matrices (there are counterexamples with eigenvalues on the unit circle). $\implies$ algorithms must be unstable 'nearby'.

(This problem was known as Van Loan's curse.)

Chu–Liu–Mehrmann algorithm

Closest thing to a solution: Chu–Liu–Mehrmann algorithm. Based on a different decomposition: $\mathcal{H} = URV^T$, with U, V orthosymplectic and

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

with R_{11}, R_{22}^* upper triangular.
Can be computed 'almost' directly in $O(n^3)$ (it's an LU-like decomposition).

Note that

$$\mathcal{H} = V \begin{bmatrix} -R_{22}^T & R_{12}^T \\ 0 & -R_{11}^T \end{bmatrix} U^T.$$

Hence

$$\mathcal{H}^2 = V \begin{bmatrix} -R_{11}R_{22}^T & * \\ 0 & -R_{22}R_{11}^T \end{bmatrix} V^T$$

can be used to compute eigenvalues and eigenvectors (for instance: the columns of V cause early breakdown in Arnoldi).