

Sign-like methods for CAREs

Matrix sign iteration

$$X_{k+1} = \frac{1}{2}(X_k + X_k^{-1}), \quad X_0 = \mathcal{H}.$$

It is not difficult to see that X_k is Hamiltonian at each step (i.e., $JX_k = -X_k^*J$). Just show that

- ▶ If M is Hamiltonian, then M^{-1} is Hamiltonian, too.
- ▶ If M_1, M_2 are Hamiltonian, then $M_1 + M_2$ is Hamiltonian, too.

(Guiding idea: Hamiltonian matrices are ‘like antisymmetric ones’: properties that you expect for antisymmetric matrices will often hold for Hamiltonian, too.)

Lemma: Se H_1, H_2 Hermitiane (i.e. $H_i^* J = -J H_i$)
allora

• $H_1 + H_2$ è Hermitiana

(basta sommare (*) per $i=1,2$)

• H_1^{-1} è Hermitiana:

$$H_1^* J = -J H_1 \rightarrow J H_1^{-1} = -H_1^* J$$

$$H^* J = -JH$$

$$J^* = -J$$

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$(JH)^* = -(H^* J) = H^* J^*$$

$$\begin{bmatrix} \textcircled{A} & \textcircled{-Q} \\ \textcircled{-G} & \textcircled{-A^T} \end{bmatrix}$$

$$JH = \begin{bmatrix} -G & -A^T \\ -A & Q \end{bmatrix}$$

$$X_{k+1} = \frac{1}{2}(X_k + X_k^{-1})$$

$$Z_k := J X_k$$

$$\boxed{Z_{k+1}} = J X_{k+1} = J \cdot \frac{1}{2}(X_k + X_k^{-1}) = \frac{1}{2}(J X_k + J X_k^{-1}) =$$

$$= \frac{1}{2}(Z_k + J Z_k^{-1} J)$$

(per calcolare Z_k^{-1} posso usare un metodo per matrici simmetriche, ad es. LDL^T , che costa la metà di un metodo per matrici generiche)

Structure-preserving sign iteration

In machine arithmetic, the X_k won't be exactly Hamiltonian — unless we modify our algorithm to ensure that they are.

Recall: X is Hamiltonian iff $Z = JX$ is symmetric.

Rewrite the iteration in terms of $Z_k := JX_k$:

$$Z_{k+1} = \frac{1}{2}(Z_k + \underbrace{JZ_k^{-1}J}), \quad Z_0 = J\mathcal{H}.$$

Will preserve symmetry exactly (assuming the method we use for inversion does).

We can incorporate scaling.

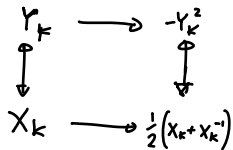
It is in some sense 'working on even pencils': given an even pencil $\lambda J - Z_k$, construct $\lambda J - Z_{k+1}$ (will see more of this idea in the following).

$J\lambda - z_k$ ha gli stessi autovalori

$$\text{di } J^{-1}z_k = X_k$$

Problema del metodo: serve invertire X_k o z_k a ogni passo. Se z_0 , o X_0 , è quasi singolare, allora posso fare errori al primo passo e non recupero più.

Towards doubling



Recall: in the ^{Sign} doubling iteration, if we set $Y_k = (I - X_k)^{-1}(I + X_k)$, then $Y_{k+1} = -Y_k^2$.

In an ideal world without rounding errors, we could compute Y_0, Y_1, Y_2, \dots , and then get the stable invariant subspace as $\ker Y_\infty$.

We can do something similar, if we work in a suitable format.

X_0 has n eigenvalues in LHP, n in RHP

Y_0 has n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with $|\lambda_i| < 1$, and n with $|\lambda_i| > 1$

$Y_k = (Y_0)^{2^k}$ has eigenvalues $\lambda_i^{2^k}$ with $\lambda_i < 1$ tending to 0 and $\lambda_i > 1$ tending to ∞

Vogliamo avere ad ogni passo

$$Y_k = \begin{bmatrix} I & G_k \\ 0 & F_k \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}$$

per opportune matrici E_k, F_k, G_k, H_k

$$Y_0 = (I - \mathcal{H})^{-1} (I + \mathcal{H})$$

Trucco: cerco una matrice $M \in \mathbb{C}^{2n \times 2n}$
invertibile tale che

$$M \begin{bmatrix} I - \mathcal{H} & \vdots & I + \mathcal{H} \\ 2n & & 2n \end{bmatrix} = \begin{bmatrix} I & G_0 & \vdots & E_0 & 0 \\ 0 & F_0 & \vdots & H_0 & I \\ 2n & & & 2n & \end{bmatrix}$$

Se vale questa identità, allora

$$M(I - \mathcal{H}) = \begin{bmatrix} I & G_0 \\ 0 & F_0 \end{bmatrix}$$

\Rightarrow

$$M(I + \mathcal{H}) = \begin{bmatrix} E & 0 \\ H_0 & I \end{bmatrix}$$

$$(I - \mathcal{H})^{-1} \cancel{M} M (I + \mathcal{H}) = \begin{bmatrix} I & G_0 \\ 0 & F_0 \end{bmatrix}^{-1} \begin{bmatrix} E_0 & 0 \\ H_0 & I \end{bmatrix}$$

Come trovare M che fa questo lavoro?

$$M \begin{bmatrix} I - A & G \\ \underline{Q} & I + A^T \end{bmatrix} = M \begin{bmatrix} I + A & -G \\ -Q & \underline{I - A^T} \end{bmatrix}$$

$$\stackrel{!}{=} \begin{bmatrix} \underline{I} & G_0 & E_0 & \underline{O} \\ \underline{O} & F_0 & H_0 & \underline{I} \end{bmatrix}$$

vale se e solo se

$$M \begin{bmatrix} I - A & -G \\ Q & I - A^T \end{bmatrix} = \underline{I}$$

(1^a, 4^a colonne)

$$e \quad M \begin{bmatrix} G & I + A \\ I + A^T & -Q \end{bmatrix} = \begin{bmatrix} G_0 & E_0 \\ F_0 & H_0 \end{bmatrix}$$

(2^a, 3^a colonne)

$$\begin{bmatrix} E_0 & G_0 \\ H_0 & F_0 \end{bmatrix} = \begin{bmatrix} I+A & -G \\ Q & I+A^* \end{bmatrix}^{-1} \begin{bmatrix} I+A & G \\ -Q & I+A^* \end{bmatrix} \quad \text{ⓧ}$$

Struttura di queste matrici:

$(I - \mathcal{H})^{-1} (I + \mathcal{H})$ è simplettica:

[parallelo del fatto che se B è antisimmetrica,
allora $(I-B)^{-1} (I+B)$ è ortogonale]

[perché B è del tipo $QA \cdot Q^T$ con Λ_i immag. puri,
e $\frac{1+x}{1-x}$ numeri $i\mathbb{R}$ nel cerchio unitario]

Lemma: se \mathcal{H} Hamiltoniana, $\mathcal{H}^* J = -J \mathcal{H}$,
allora $S = (1 - \mathcal{H})^{-1} (1 + \mathcal{H})$ è simplettica, cioè

$$S^* J S = J$$

$$(1 - \mathcal{H})^{-*} (1 + \mathcal{H}^*) J (1 + \mathcal{H}) (1 - \mathcal{H})^{-1} \stackrel{?}{=} J$$

$$(1 + \mathcal{H})^* J (1 + \mathcal{H}) \stackrel{?}{=} (1 - \mathcal{H})^* J (1 - \mathcal{H})$$

~~$$J + \mathcal{H}^* J + J \mathcal{H} + \mathcal{H}^* J \mathcal{H} \stackrel{?}{=} J - \mathcal{H}^* J - J \mathcal{H} + \mathcal{H}^* J \mathcal{H}$$~~

$$2\mathcal{H}^* J \stackrel{?}{=} -2J\mathcal{H}$$

✓ perché equivale a \mathcal{H} Hamiltoniana

(Volendo, è un risultato su pencil:

Def. Una pencil $\lambda E - A$ è simplettica se $E J E^* = A J A^*$

$$\Leftrightarrow (E^{-1}A)^* J (E^{-1}A) = J$$

Quando è che $\begin{bmatrix} 1 & G_0 \\ 0 & F_0 \end{bmatrix}^{-1} \begin{bmatrix} E_0 & 0 \\ H_0 & 1 \end{bmatrix}$ è simplettica?

Quando

$$\begin{bmatrix} E_0 & 0 \\ H_0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} E_0 & 0 \\ H_0 & 1 \end{bmatrix}^* = \begin{bmatrix} 1 & G_0 \\ 0 & F_0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & G_0 \\ 0 & F_0 \end{bmatrix}^*$$

$$\begin{bmatrix} 0 & E_0 \\ -1 & H_0 \end{bmatrix} \begin{bmatrix} E_0^* & H_0^* \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -G_0 & 1 \\ -F_0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ G_0^* & F_0^* \end{bmatrix}$$

$$\begin{bmatrix} 0 & E_0 \\ -E_0^* & -H_0^* + H_0 \end{bmatrix} = \begin{bmatrix} -G_0 + G_0^* & F_0^* \\ -F_0 & 0 \end{bmatrix}$$

$$\Leftrightarrow G_0 = G_0^*, \quad H_0 = H_0^*, \quad E_0 = F_0^*$$

$$(\mathcal{E}^{-1}A)^* J (\mathcal{E}^{-1}A) = J$$

$$A^* \mathcal{E}^* J \mathcal{E}^{-1} A = J$$

$$\mathcal{E}^{-*} J \mathcal{E}^{-1} = A^{-*} J A^{-1}$$

$$\boxed{\mathcal{E} J \mathcal{E}^* = A J A^*}$$

Standard Symplectic Form

Goal: write $Y_0 = (I - \mathcal{H})^{-1}(I + \mathcal{H})$ as

$$Y_0 = \begin{bmatrix} I & -G_0 \\ 0 & F_0 \end{bmatrix}^{-1} \begin{bmatrix} E_0 & 0 \\ -H_0 & I \end{bmatrix}.$$

Can be rewritten as: find M such that

$$M \begin{bmatrix} (I - \mathcal{H}) & (I + \mathcal{H}) \end{bmatrix} = \begin{bmatrix} I & G_0 & E_0 & 0 \\ 0 & F_0 & H_0 & I \end{bmatrix}.$$

Solution: M is the inverse of block columns (1, 4).

Structural properties: if \mathcal{H} is Hamiltonian, Y_0 is symplectic. If Y_0 is symplectic, $E_0 = F_0^*$, $G_0 = G_0^*$, $H_0 = H_0^*$.

Moreover, if $G \succeq 0$, $H \succeq 0$, then $G_0 \succeq 0$, $H_0 \preceq 0$.

(All these manipulations can be reformulated as left-multiplication of the pencil $(I - \mathcal{H}, I + \mathcal{H})$.)

$$Y_0 \rightarrow Y_1 = -Y_0^2$$

$$Y_k \rightarrow Y_{k+1} = -Y_k^2$$

$$\begin{matrix} \parallel \\ \left[\begin{array}{c|c} 1 & G_{k+1} \\ \hline 0 & F_{k+1} \end{array} \right]^{-1} \left[\begin{array}{c|c} E_{k+1} & 0 \\ \hline H_{k+1} & 1 \end{array} \right] = - \left(\left[\begin{array}{c|c} 1 & G_k \\ \hline 0 & F_k \end{array} \right]^{-1} \left[\begin{array}{c|c} E_k & 0 \\ \hline H_k & 1 \end{array} \right] \right)^2 \\ \underbrace{\quad\quad\quad}_{M_{k+1}^{-1}} \quad \underbrace{\quad\quad\quad}_{N_{k+1}} \quad \underbrace{\quad\quad\quad}_{M_k^{-1}} \quad \underbrace{\quad\quad\quad}_{N_k} \end{matrix}$$

Visto come mappa

$$(E_k, F_k, G_k, H_k) \rightarrow (E_{k+1}, F_{k+1}, G_{k+1}, H_{k+1})$$

Abbiamo M_k, N_k , vogliamo fattorizzare

$$M_{k+1}^{-1} N_{k+1} = - \left(M_k^{-1} N_k \right)^2 = - M_k^{-1} N_k \overbrace{M_k^{-1} N_k}^{\leftarrow \rightarrow}$$

Se troviamo \hat{M}_k, \hat{N}_k tali che $N_k M_k^{-1} = \hat{M}_k^{-1} \hat{N}_k$,

Abbiamo

$$\begin{aligned} -m_k^{-1} \underbrace{n_k m_k^{-1}}_{n_k} &= -m_k^{-1} \underbrace{\hat{m}_k^{-1} \hat{n}_k}_{\hat{n}_k} n_k = \\ &= \underbrace{(-\hat{m}_k m_k)^{-1}}_{m_{k+1}^{-1}} \underbrace{(\hat{n}_k n_k)}_{n_{k+1}}. \end{aligned}$$

$$n_k m_k^{-1} = \hat{m}_k^{-1} \hat{n}_k \iff \hat{m}_k n_k = \hat{n}_k m_k$$

$$\iff \begin{bmatrix} -\hat{m}_k & \hat{n}_k \end{bmatrix} \begin{bmatrix} n_k \\ m_k \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} 1 & \hat{G}_k & \dots & \hat{E}_k & 0 \\ 0 & \hat{F}_k & \dots & \hat{H}_k & 1 \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & 1 \\ 1 & G_k \\ 0 & F_k \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & \dots & \hat{E}_k & \hat{G}_k \\ 0 & 1 & \dots & \hat{H}_k & \hat{F}_k \end{bmatrix} \begin{bmatrix} E_k & 0 \\ 0 & F_k \\ \dots & \dots \\ 1 & G_k \\ H_k & 1 \end{bmatrix} = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix} + \begin{bmatrix} \hat{E}_k & \hat{G}_k \\ \hat{H}_k & \hat{F}_k \end{bmatrix} \begin{bmatrix} 1 & G_k \\ H_k & 1 \end{bmatrix} = 0$$

$4n \times 2n$, ci aspettiamo
 che il suo kernel sia
 abbia dimensione $2n$,
 quindi di poter trovare
 una matrice con $2n$
 vettore lin. indipendenti.

$$\begin{aligned}
 \begin{pmatrix} \hat{E}_k & \hat{G}_k \\ \hat{H}_k & \hat{F}_k \end{pmatrix} &= - \begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix} \begin{bmatrix} I & G_k \\ H_k & I \end{bmatrix}^{-1} = \\
 &= - \begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix} \cdot \begin{bmatrix} (I - G_k H_k)^{-1} & -G_k (I - H_k G_k)^{-1} \\ -H_k (I - G_k H_k)^{-1} & (I - H_k G_k)^{-1} \end{bmatrix} \\
 &= \begin{bmatrix} -E_k (I - G_k H_k)^{-1} & E_k G_k (I - H_k G_k)^{-1} \\ F_k H_k (I - G_k H_k)^{-1} & -F_k (I - H_k G_k)^{-1} \end{bmatrix}.
 \end{aligned}$$

$$\begin{bmatrix} I & G_k \\ H_k & I \end{bmatrix} \cdot \begin{bmatrix} I & -G_k \\ -H_k & I \end{bmatrix} = \begin{bmatrix} I - G_k H_k & 0 \\ 0 & I - H_k G_k \end{bmatrix}$$

$$M_{k+1} = (-\hat{M}_k) M_k = \begin{bmatrix} I & \hat{G}_k \\ 0 & \hat{F}_k \end{bmatrix} \begin{bmatrix} I & G_k \\ 0 & F_k \end{bmatrix} =$$

$$= \begin{bmatrix} I & E_k G_k (I - H_k G_k)^{-1} \\ 0 & -F_k (I - H_k G_k)^{-1} \end{bmatrix} \begin{bmatrix} I & G_k \\ 0 & F_k \end{bmatrix} = \begin{bmatrix} I & \\ 0 & \end{bmatrix}$$

$$\begin{bmatrix} G_k + E_k G_k (I - H_k G_k)^{-1} F_k \\ -F_k (I - H_k G_k)^{-1} F_k \end{bmatrix}$$

$$N_{k+1} = \hat{N}_k N_k = \begin{bmatrix} \hat{E}_k & 0 \\ \hat{H}_k & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} =$$

$$= \begin{bmatrix} -E_k (I - G_k H_k)^{-1} & 0 \\ F_k H_k (I - G_k H_k)^{-1} & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} = \begin{bmatrix} -E_k (I - G_k H_k)^{-1} E_k & 0 \\ H_k + F_k H_k (I - G_k H_k)^{-1} E_k & I \end{bmatrix}$$

E_{k+1}
!!

ii
 H_{k+1}

G_{k+1}
!!

ii
 F_{k+1}

Structured doubling algorithm

$$\begin{bmatrix} E_0 & G_0 \\ H_0 & F_0 \end{bmatrix} = \begin{bmatrix} I-A & -G \\ Q & I-A^T \end{bmatrix}^{-1} \begin{bmatrix} I+A & G \\ Q & I+A^T \end{bmatrix}.$$

for $k=0, 1, 2, \dots$

$$\begin{cases} E_{k+1} = -E_k (I - G_k H_k)^{-1} E_k \\ F_{k+1} = -F_k (I - H_k G_k)^{-1} F_k \\ G_{k+1} = G_k + E_k G_k (I - H_k G_k)^{-1} F_k \\ H_{k+1} = H_k + F_k H_k (I - G_k H_k)^{-1} E_k \end{cases}$$

scambiare

$$E_k \leftrightarrow F_k$$

$$G_k \leftrightarrow H_k$$

le due
invarianta

end

Per come le abbiamo derivate, X_k dell'it. segno soddisfa

$$(I - X_k)^{-1} (I + X_k) = \begin{bmatrix} I & G_k \\ 0 & F_k \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}$$

Struttura: ad ogni passo abbiamo:

$$(1) E_k = F_k^* \quad (2) G_k = G_k^* \quad (3) H_k = H_k^* \quad (\text{simmetticit\`a})$$

$$\text{e anche } 0 \preceq G_0 \preceq G_1 \preceq G_2 \preceq \dots \quad (4)$$

$$0 \succeq H_0 \succeq H_1 \succeq H_2 \succeq \dots \quad (5)$$

dim: Per induzione

$$(1): \text{Facile, basta fare } F_{k+1}^* = F_k^* (1 - H_k G_k)^{-*} F_k^* = \\ = E_k (1 - G_k H_k)^{-1} E_k$$

(2) Ci basta dire che $G_k (1 - H_k G_k)^{-1}$ \u00e8 simmetrica

Se posso espandere la serie, vale

$$G_k (1 - H_k G_k)^{-1} = G_k + G_k H_k G_k + G_k H_k G_k H_k G_k + \dots \\ = (1 - G_k H_k)^{-1} G_k$$

Anche se non posso espandere la serie,

$$\text{vale } G_K (I - H_K G_K)^{-1} = (I - G_K H_K)^{-1} G_K$$

$$(\bar{\text{è un'identità: }} (I - G_K H_K) G_K = G_K (I - H_K G_K))$$

Questo mi dice anche che

$$(G_K (I - H_K G_K)^{-1})^* = (I - G_K H_K)^{-1} G_K = G_K (I - H_K G_K)^{-1}$$

è simmetrica

(3) è uguale a (2)

(4) Mi basta dire che $G_K (I - H_K G_K)^{-1} \succeq 0$

Posso sempre scrivere $G_K = R_K R_K^*$, e ho

$$G_K (I - H_K G_K)^{-1} = R_K (I - R_K^* H_K R_K)^{-1} R_K^* .$$

Se $H_K \succcurlyeq 0$, allora

$$R_K \left(I - \underbrace{R_K^* H_K R_K}_{\succeq 0} \right)^{-1} R_K^* \succeq 0$$

(5) uguale a (4).

[Nota che le inversioni che servono si riescono a fare, perché $R_K^* H_K R_K \succcurlyeq 0$, quindi $I + R_K^* H_K R_K \succ 0$, e similmente anche $I - G_K H_K$
 $I - H_K G_K$

non sono mai singolari (

($G_K H_K$ ha gli stessi autoval. di $H_K G_K$
e di $R_K^* H_K R_K$, che sono tutti ≤ 0 .)

Costo: l'eq. per F_{k+1} non serve (è il trasposto di F_{k+1}). Le inversioni si riducono a una sola, perché $I - H_K G_K = (I - G_K H_K)^*$, e anzi con un po' di attenzione ci si riduce a invertire una matrice simmetrica.

Costo \sim comparabile con l'iterat. sequo
(in cui c'è da invertire una matrice $2n \times 2n$)

Doubling algorithm

Plan Given $Y_k = \begin{bmatrix} I & G_k \\ 0 & E_k^* \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}$, compute

$$Y_{k+1} = -Y_k^2 = \begin{bmatrix} I & G_{k+1} \\ 0 & E_{k+1}^* \end{bmatrix}^{-1} \begin{bmatrix} E_{k+1} & 0 \\ H_{k+1} & I \end{bmatrix}.$$

Similar to the 'inverse-free sign method' described earlier.

If $Y_k = \mathcal{M}_k^{-1} \mathcal{N}_k$, then $-Y_k^2 = -\mathcal{M}_k^{-1} \mathcal{N}_k \mathcal{M}_k^{-1} \mathcal{N}_k = \mathcal{M}_k^{-1} \widehat{\mathcal{M}}_k^{-1} \widehat{\mathcal{N}}_k \mathcal{N}_k = (\widehat{\mathcal{M}}_k \mathcal{M}_k)^{-1} (\widehat{\mathcal{N}}_k \mathcal{N}_k)$, where $\widehat{\mathcal{M}}_k, \widehat{\mathcal{N}}_k$ satisfy $\widehat{\mathcal{M}}_k^{-1} \widehat{\mathcal{N}}_k = -\mathcal{N}_k \mathcal{M}_k^{-1}$, i.e.,

$$\begin{bmatrix} \widehat{\mathcal{M}}_k & \widehat{\mathcal{N}}_k \end{bmatrix} \begin{bmatrix} \mathcal{N}_k \\ \mathcal{M}_k \end{bmatrix} = 0.$$

Doubling: inversion trick

$$\begin{bmatrix} I & \widehat{G}_k & \widehat{E}_k & 0 \\ 0 & \widehat{F}_k & \widehat{H}_k & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \\ I & G_k \\ 0 & E_k^* \end{bmatrix} = 0$$

holds if

$$\begin{aligned} \begin{bmatrix} \widehat{G}_k & \widehat{E}_k \\ \widehat{F}_k & \widehat{H}_k \end{bmatrix} &= - \begin{bmatrix} E_k & 0 \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} H_k & I \\ I & G_k \end{bmatrix}^{-1} \\ &= \begin{bmatrix} E_k & 0 \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} G_k(I - H_k G_k)^{-1} & -(I - G_k H_k)^{-1} \\ -(I - H_k G_k)^{-1} & H_k(I - G_k H_k)^{-1} \end{bmatrix}. \end{aligned}$$

Doubling: the formulas

Putting everything together,

$$\begin{aligned}\begin{bmatrix} E_{k+1} & 0 \\ H_{k+1} & I \end{bmatrix} &= \begin{bmatrix} -E_k(I - G_k H_k)^{-1} & 0 \\ E_k^* H_k (I - G_k H_k)^{-1} & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} \\ &= \begin{bmatrix} -E_k(I - G_k H_k)^{-1} E_k & 0 \\ H_k + E_k^* H_k (I - G_k H_k)^{-1} E_k & I \end{bmatrix}\end{aligned}$$

and an analogous computation gives E_{k+1}^* , G_{k+1} :

Structured doubling algorithm

$$\begin{aligned}E_{k+1} &= -E_k(I - G_k H_k)^{-1} E_k, \\ G_{k+1} &= G_k + E_k G_k (I - H_k G_k)^{-1} E_k^*, \\ H_{k+1} &= H_k + E_k^* H_k (I - G_k H_k)^{-1} E_k.\end{aligned}$$

SDA: details

Note that (even when the middle term does not converge)

$$G_k(I - H_k G_k)^{-1} = G_k + G_k H_k G_k + G_k H_k G_k H_k G_k + \dots = (I - G_k H_k)^{-1} G_k,$$

and this matrix is symmetric. If $G_k = B_k B_k^*$, then it can also be rewritten as $B_k(I - B_k^* H_k B_k)^{-1} B_k^*$ (inverting a symmetric matrix).

Monotonicity If $H_k \preceq 0$ then $G_k(I - H_k G_k)^{-1} \succeq 0$. Hence, $0 \preceq G_0 \preceq G_1 \preceq \dots$, and $0 \succeq H_0 \succeq H_1 \succeq H_2 \succeq \dots$

Cost As much as a $2n \times 2n$ inversion $M^{-1}N$, if you put everything together. Unlike the sign algorithm, we have a bound $\sigma_{\min}(I - H_k G_k) \geq 1$.

SDA: the dual equation

To analyze convergence, we need to introduce another matrix. Let Y be the matrix such that

$$\mathcal{H} \begin{bmatrix} -Y \\ I \end{bmatrix} = \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} -Y \\ I \end{bmatrix} = \begin{bmatrix} -Y \\ I \end{bmatrix} \hat{\mathcal{R}}$$

is the **anti-stable** invariant subspace of \mathcal{H} , i.e., $\Lambda(\hat{\mathcal{R}}) \subset RHP$.

Why does it exist? Because $\begin{bmatrix} I \\ Y \end{bmatrix}$ spans the stable subspace of $\mathcal{H}^* = -J\mathcal{H}J$, and we can repeat our arguments on it (in the version with (A, B) and (A^T, C^T) controllable)

SDA: convergence (intuitively)

Intuitive view E_k approximately squared at each time. Hence

$$\mathcal{H}_k = \begin{bmatrix} I & G_k \\ 0 & E_k^* \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix}$$

has n eigenvalues $\rightarrow 0$ and n that $\rightarrow \infty$. $\ker \mathcal{H}_k \approx \begin{bmatrix} I \\ -H_k \end{bmatrix}$, so
 $-H_k \rightarrow X$.

Dually, “ $\ker \mathcal{H}_k^{-1}$ ” (a thing that shouldn't exist...) $\approx \begin{bmatrix} -G_k \\ I \end{bmatrix}$, so
 $G_k \rightarrow Y$.

Convergenza: $E_k \rightarrow 0$

$H_k \rightarrow -X$ (soluzione della CANE)
stabilizzante

$G_k \rightarrow Y$

dove Y è la matrice tale che

$\mathcal{H} \begin{bmatrix} Y \\ I \end{bmatrix} = \begin{bmatrix} Y \\ I \end{bmatrix} \hat{R}$ è il sottosp. invariante
anti-stabile (associato agli autoval. nel RHP)

La convergenza è quadratica.

(Y esiste per lo stesso motivo per cui $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ x \end{bmatrix}$, su $J\mathcal{H}J$).

$$\mathcal{H}\left[\begin{matrix} 1 \\ x \end{matrix}\right] = \begin{bmatrix} 1 \\ x \end{bmatrix} R \quad \wedge(R) \subseteq \text{LHP}$$

$$Y_0 \begin{bmatrix} 1 \\ x \end{bmatrix} = (1 - \mathcal{H}_0)^{-1} (1 + \mathcal{H}_0) \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix} (1 - R)^{-1} (1 + R)$$

dove $(1 - R)^{-1} (1 + R) = S$ ha autoval. nel
cerchio unitario, $\rho(S) < 1$

$$Y_k \begin{bmatrix} 1 \\ x \end{bmatrix} = -(Y_0)^{2^k} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix} S^{2^k}$$

$$\begin{bmatrix} 1 & G_k \\ 0 & F_k \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ H_k & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = - \begin{bmatrix} 1 \\ x \end{bmatrix} S^{2^k} \quad \longrightarrow \ominus$$

$$\begin{pmatrix} E_k & 0 \\ H_k & 1 \end{pmatrix} \begin{bmatrix} 1 \\ X \end{bmatrix} = \begin{bmatrix} 1 & G_k \\ 0 & f_k \end{bmatrix} \begin{bmatrix} 1 \\ X \end{bmatrix} (-S^{2k})$$

$$\begin{cases} E_k = (1 + G_k X) (-S^{2k}) \end{cases}$$

$$\begin{cases} H_k + X = F_k X (-S) = E_k^* X (-S) = \\ = (-S^*)^{2k} \end{cases}$$

SDA convergence (formally)

More formally

$$\mathcal{H}_0 \begin{bmatrix} I \\ X \end{bmatrix} = (I - \mathcal{H})^{-1}(I + \mathcal{H}) \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (I - \mathcal{R})^{-1}(I + \mathcal{R}).$$

where $\mathcal{S} = (I - \mathcal{R})^{-1}(I + \mathcal{R})$ has all eigenvalues in the unit circle.

$$\begin{bmatrix} I & G_k \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \begin{bmatrix} E_k & 0 \\ H_k & I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} \mathcal{S}^{2^k}.$$

which implies

$$\begin{aligned} E_k &= (I + G_k X) \mathcal{S}^{2^k}, \\ H_k + X &= E_k^* X \mathcal{S}^{2^k} = (\mathcal{S}^{2^k})^* (I + X G_k) \mathcal{S}^{2^k} \succeq 0. \end{aligned}$$

The same computation on the dual equation gives $G_k \preceq Y$, so G_k is bounded and $E_k \rightarrow 0, H_k + X \rightarrow 0$ (quadratically as \mathcal{S}^{2^k}).